

Quantum metrology with nonequilibrium steady states of quantum spin chains

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We consider parameter estimations with probes being the boundary-driven or dissipated nonequilibrium steady states of XXZ spin-1/2 chains. The parameters to be estimated are the dissipation coupling and the anisotropy of the spin-spin interaction. In the weak-coupling regime we compute the scaling of the Fisher information, i.e., the inverse best sensitivity among all estimators, with the number of spins. We find superlinear scalings and transitions between the distinct isotropic and anisotropic phases. We also look at the best relative error, which decreases with the number of particles faster than the shot-noise only for the estimation of anisotropy.

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I. INTRODUCTION

Metrology, i.e., the ability to perform precise measurements, is key for technological and experimental developments. Among the most advanced metrological schemes are thermometry [1–10], magnetometry [11–18], and interferometry [19–37]. Recently, estimations of Hamiltonian parameters have also been suggested [38–41]. In the quantum domain, either thermal equilibrium or pure states have been employed most often. Moreover, real experiments are always affected by unavoidable noise due to the interaction with the environment, which can be described by Markovian master equations under controllable approximations [42,43]. Dissipation has been shown to be greatly detrimental for interferometry [44–47]. On the other hand, estimations of parameters of noisy dynamics have been proposed in [48–54].

In the present paper, we consider a spin-1/2 chain with XXZ spin interaction driven by local noise at its ends. The unique asymptotic state of the corresponding master equation has been recently derived in terms of matrix product operators [55–61], resulting in a nonequilibrium steady state (NESS). Such a NESS depends on the parameters describing the dynamics and the properties of the environments at the ends of the chain. Based on this dependence, we propose to use the NESS as a probe to estimate the above parameters.

We quantify the performance of the parameter estimations with the Fisher information [62–65], which gives principally the best achievable sensitivity. It is typically important to study the scaling of the Fisher information with the number of resources: the faster the Fisher information grows, either smaller devices are required for constant sensitivities, or more precise estimations can be performed at fixed sizes. Most frequently, classical devices show a linear scaling of the Fisher information and thus a linear decrease of the best absolute sensitivity, known as shot noise. We find superlinear scalings of the Fisher information with respect to different parameters and in different regimes. In particular, we find a phase transition between power-law scaling of Fisher information in the regime of easy-plane interactions to superexponential scaling in the regime of easy-axis interactions for the perturbative range of environment coupling. Within this perturbative analysis, we need to consider the relative error, which decreases faster than linearly in the particle number only for the estimation of the anisotropy parameter for isotropic and easy-plane interactions. In the latter case, the rate of growth of Fisher information is

a nowhere continuous function of the anisotropy parameter anywhere.

II. THE SYSTEM

We focus on the following Markovian dissipative dynamics of one-dimensional n -spin chains:

$$\frac{d}{dt}\rho(t) = -i\left[\frac{\Omega}{2}M_z + JH_{XXZ}, \rho(t)\right] + \lambda \sum_{k=1}^4 \left(L_k \rho(t) L_k^\dagger - \frac{1}{2} \{ L_k^\dagger L_k, \rho(t) \} \right), \quad (1)$$

where $M_z = \sum_{j=1}^n \sigma_j^z$ is the total magnetization along the z direction,

$$H_{XXZ} = \sum_{j=1}^{n-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z) \quad (2)$$

is the Hamiltonian of the XXZ spin chain,

$$L_{1,2} = \sqrt{\frac{1 \pm \mu}{2}} \sigma_1^\pm, \quad L_{3,4} = \sqrt{\frac{1 \mp \mu}{2}} \sigma_N^\pm \quad (3)$$

are the Lindblad noise driving channels, and σ_j^α are the Pauli matrices of the j th spin. Master equations with local Lindblad operators, i.e., with each environment interacting with a single particle as in (1), can be derived from microscopic models of system-environment interaction following [66] provided $\Omega \gg J$; a concrete example was studied in [67]. The local Hamiltonian generator $-i[M_z, \cdot]$ commutes with the other terms in (1), namely, $-i[H_{XXZ}, \cdot]$ and $L_k \cdot L_k^\dagger - \frac{1}{2} \{ L_k^\dagger L_k, \cdot \}$. Therefore, the NESS $\rho_\infty \equiv \lim_{t \rightarrow \infty} \rho(t)$ does not depend on the presence of the generator $-i[M_z, \cdot]$ and formally equals the NESS derived in the absence of such a generator [55,56,60], which is unique:

$$\rho_\infty = 2^{-n} \left(\mathbb{1} + i \frac{\lambda}{2J} \mu (Z - Z^\dagger) + \frac{\lambda^2}{8J^2} \{ \mu [Z, Z^\dagger] - \mu^2 (Z - Z^\dagger)^2 \} \right) + O\left(\frac{\lambda}{J}\right)^3 \quad (4)$$

(see [57] for a generalization to the case of asymmetric driving and [61] for a review). Here Z is a matrix product operator

$$Z = \sum_{\{s_1, \dots, s_N\} \in \{0, +, -\}^N} \langle L | \prod_{j=1}^n A_{s_j} | R \rangle \bigotimes_{j=1}^n \sigma_j^{s_j}, \quad (5)$$

with tridiagonal matrices A_{s_j} on the auxiliary Hilbert space \mathcal{H} spanned by the orthonormal basis $\{|L\rangle, |R\rangle, |1\rangle, |2\rangle, \dots, |\lfloor \frac{n}{2} \rfloor\rangle\}$:

$$\begin{aligned} A_0 &= |L\rangle\langle L| + |R\rangle\langle R| + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \cos(\eta k) |k\rangle\langle k|, \\ A_+ &= |1\rangle\langle R| - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \sin(\eta k) |k+1\rangle\langle k|, \\ A_- &= |L\rangle\langle 1| + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \sin[\eta(k+1)] |k\rangle\langle k+1|, \end{aligned} \quad (6)$$

and $\eta = \arccos \Delta \in \mathbb{R} \cup i\mathbb{R}$. The expansion (4) holds as soon as the zeroth order is larger than the first order in $\frac{\lambda}{J}$. Estimating the magnitude of each order with its Hilbert-Schmidt norm [$\|O\|_{\text{HS}} = \sqrt{\text{Tr}(OO^\dagger)}$], the validity condition for (4) reads

$$\frac{\lambda}{J} < \frac{\sqrt{2^{n+1}}}{\mu} \|Z\|_{\text{HS}}^{-1}. \quad (7)$$

If $\mu = 1$, the nonperturbative NESS at any order of $\frac{\lambda}{J}$ [56,60] is

$$\rho_\infty = \frac{SS^\dagger}{\text{Tr}(SS^\dagger)}, \quad S = \sum_{\{s_1, \dots, s_n\} \in \{0, +, -\}^n} \langle 0 | \prod_{j=1}^n B_{s_j} | 0 \rangle \bigotimes_{j=1}^n \sigma_j^{s_j}, \quad (8)$$

with the matrix product operator S and tridiagonal matrices B_{s_j} on the auxiliary Hilbert space spanned by the orthonormal basis $\{|0\rangle, |1\rangle, |2\rangle, \dots, |\lfloor \frac{n}{2} \rfloor\rangle\}$,

$$\begin{aligned} B_0 &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sin[\eta(s-k)] |k\rangle\langle k|, \\ B_+ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sin[\eta(k-2s)] |k\rangle\langle k+1|, \\ B_- &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sin[\eta(k+1)] |k+1\rangle\langle k|, \end{aligned} \quad (9)$$

with s given by $\cot(s\eta) = \frac{\epsilon}{4i \sin \eta}$.

III. QUANTUM ESTIMATION THEORY

In this section, we discuss some fundamental aspects of quantum metrology [62–65] that are relevant in our study. We are interested in estimating one of the parameters x of the NESS, e.g., $\{J, \Delta, \lambda, \mu\}$. We can express the change of the NESS with respect to x by the following equation:

$$\frac{d}{dx} \rho_\infty = \frac{1}{2} (L_x \rho_\infty + \rho_\infty L_x), \quad (10)$$

where

$$L_x = 2 \int_0^\infty ds e^{-s\rho_\infty} \frac{d\rho_\infty}{dx} e^{-s\rho_\infty} \quad (11)$$

is the symmetric logarithmic derivative, which can, in general, depend on the parameter x to be estimated. Quantum estimation theory gives the best achievable sensitivities, namely, estimator variances, through the quantum Cramér-Rao bound [62–65],

$$\delta^2 x \geq \frac{1}{F_x}, \quad (12)$$

where

$$\begin{aligned} F_x &= \text{Tr}(L_x^2 \rho_\infty) = \text{Tr}\left(L_x \frac{d\rho_\infty}{dx}\right) \\ &= 2 \int_0^\infty ds \text{Tr}\left[\left(\frac{d\rho_\infty}{dx} e^{-s\rho_\infty}\right)^2\right] \end{aligned} \quad (13)$$

is called Fisher information. The quantum Cramér-Rao bound (12) can be saturated by a projective measurement onto the eigenstates of the symmetric logarithmic derivative. This measurement may, however, depend on the parameter to be estimated and thus might not be practically relevant. On the one hand, it is an open problem to find practical and optimal estimations; on the other hand, Fisher information itself provides the theoretical bound of the estimation sensitivity. The best relative errors are thus

$$\min_{\text{all estimations}} \frac{\delta x}{x} = \frac{1}{x\sqrt{F_x}}. \quad (14)$$

In the multiparameter estimation the inverse of the covariance matrix of any estimation is bounded from below by the matrix $[F_{x,x'}]_{x,x'}$, with $F_{x,x'} = \frac{1}{2} \text{Tr}(\rho_\infty \{L_x, L_{x'}\})$. Thus, $1/F_x$ is still the best variance for the estimation of x , and the covariances depend on $F_{x,x'}$. Nevertheless, the matrix bound is not saturable in general because the optimal estimation of different parameters does not commute in general and thus cannot be simultaneously performed. A tight multiparameter bound is still an open problem.

We compute the Fisher information (13) for the nonperturbative NESS ($\mu = 1$) numerically and its leading order for the perturbative NESS (4) analytically. At the lowest order in $\frac{\lambda}{J}$, the Fisher information is $F_x = F_x^{(0)} + O(\frac{\lambda}{J})^\nu$, where $\nu = 2$ if $x \equiv \lambda$ and $\nu = 4$ otherwise, and

$$\begin{aligned} F_x^{(0)} &= 2 \int_0^\infty ds e^{-\frac{2s}{J}} \text{Tr}\left[\left(\frac{d}{dx} \frac{i\lambda\mu}{J 2^{n+1}} (Z - Z^\dagger)\right)^2\right] \\ &= \frac{1}{2^{n+1}} \left\| \frac{d}{dx} \frac{\lambda\mu Z}{J} \right\|_{\text{HS}}^2. \end{aligned} \quad (15)$$

Note that the contributions of order $(\frac{\lambda}{J})^{\nu-1}$ vanish because they are traces of real, anti-Hermitian matrices, as can be realized by plugging (4) into (13). Only the parameter Δ enters nontrivially in the operator Z , whereas the others enter as multiplicative constants. Henceforth, we consider the case of parameter Δ separately, provided the latter is independent of the other parameters.

A. Computation of $F_x^{(0)}$ with $x \neq \Delta$

In this section we explicitly compute $F_x^{(0)}$ for x different from Δ . Plugging (5) and (6) into (15), we get

$$\begin{aligned}
 F_x^{(0)} &= \frac{1}{2^{n+1}} \left(\frac{d}{dx} \frac{\lambda\mu}{J} \right)^2 \sum_{\substack{\{s_1, \dots, s_n\}, \{s'_1, \dots, s'_n\} \\ \in \{0, +, -\}^n}} \\
 &\times \langle L | \prod_{j=1}^n A_{s_j} | R \rangle \langle L | \prod_{j=1}^n A_{s'_j} | R \rangle \text{Tr} \left[\bigotimes_{j=1}^n \sigma_j^{s_j} (\sigma_j^{s'_j})^\dagger \right] \\
 &= \frac{1}{2} \left(\frac{d}{dx} \frac{\lambda\mu}{J} \right)^2 \langle L | \otimes \langle L | \\
 &\times \left(A_0 \otimes A_0 + \frac{1}{2} A_+ \otimes A_+ + \frac{1}{2} A_- \otimes A_- \right)^n | R \rangle \otimes | R \rangle.
 \end{aligned} \tag{16}$$

Since the subspace $\{|k\rangle \otimes |k\rangle\}_{k=L, R, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor}$ of the auxiliary space is preserved by operators $A_{s_j} \otimes A_{s'_j}$, we apply the mapping $|k\rangle \otimes |k\rangle \rightarrow |k\rangle$ and finally obtain

$$F_x^{(0)} = \left(\frac{d}{dx} \frac{\lambda\mu}{J} \right)^2 \frac{\langle L | T^n | R \rangle}{2} \quad \forall x \text{ except } \Delta, \tag{17}$$

where the transfer matrix T is the following:

$$\begin{aligned}
 T &= |L\rangle \langle L| + |R\rangle \langle R| + \frac{|L\rangle \langle 1| + |1\rangle \langle R|}{2} \\
 &+ \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(\cos^2(\eta k) |k\rangle \langle k| + \frac{\sin^2(\eta k)}{2} |k+1\rangle \langle k| \right. \\
 &\left. + \frac{\sin^2[\eta(k+1)]}{2} |k\rangle \langle k+1| \right).
 \end{aligned} \tag{18}$$

The smallest-order Fisher information $F_x^{(0)}$ is then expressed in terms of the exponential of the transfer matrix T . This form enables the computation of $F_x^{(0)}$ using the combinatorics of n -step paths going from $|R\rangle$ to $|L\rangle$ by means of climbs and descents through intermediate states $\{|k\rangle\}_{k \geq 1}$. The amplitude of each step is the corresponding matrix element of the transfer matrix T .

The optimal estimator of any parameter x except Δ , attaining the quantum Cramér-Rao bound (12) at the lowest order in $\frac{\lambda}{J}$, is devised by a measurement $\text{Tr}(\zeta \rho_\infty)$ of an observable given by Hermitian operator $\zeta \equiv i(Z - Z^\dagger)$. The experimental input is $\zeta_m = \frac{1}{m} \sum_{j=1}^m z_j$, where $\{z_j\}_{j=1, \dots, m}$ are m measurement outcomes of ζ . Each value z_j is sampled from the probability distribution of measuring such a value from a system in the state ρ_∞ [68]. The average and the variance of ζ_m with respect to this probability distribution are

$$\begin{aligned}
 \mathbb{E}[\zeta_m] &= \text{Tr}(\zeta \rho_\infty), \\
 \text{Var} \zeta_m &\equiv \mathbb{E}\{[\zeta_m - \text{Tr}(\zeta \rho_\infty)]^2\} \\
 &= \frac{1}{m} [\text{Tr}(\zeta^2 \rho_\infty) - (\text{Tr} \zeta \rho_\infty)^2].
 \end{aligned} \tag{19}$$

For large m , $\zeta_m \rightarrow \text{Tr}(\zeta \rho_\infty)$ is the statistical average of experimental outcomes which converges to the expectation of the operator ζ . Inverting the relation $\zeta_m = \text{Tr}(\zeta \rho_\infty)$, we

estimate x with sensitivity at the lowest order in $\frac{\lambda}{J}$,

$$\delta^2 x = \frac{\text{Var} \zeta_m}{\left[\frac{d}{dx} \text{Tr}(\zeta \rho_\infty) \right]^2} = \frac{1}{m F_x^{(0)}} \quad \forall x \text{ except } \Delta. \tag{20}$$

B. Computation of $F_\Delta^{(0)}$

We now derive an explicit formula for $F_\Delta^{(0)}$. With expressions (5) and (6), Eq. (15) for $x \equiv \Delta$ becomes

$$\begin{aligned}
 F_\Delta^{(0)} &= \frac{1}{2^{n+1}} \left(\frac{\lambda\mu}{J} \right)^2 \sum_{\substack{\{s_1, \dots, s_n\}, \{s'_1, \dots, s'_n\} \\ \in \{0, +, -\}^n}} \langle L | \frac{d}{d\Delta} \\
 &\times \left(\prod_{j=1}^n A_{s_j} \right) | R \rangle \langle L | \frac{d}{d\Delta} \left(\prod_{j=1}^n A_{s'_j} \right) | R \rangle \\
 &\times \text{Tr} \left[\bigotimes_{j=1}^n \sigma_j^{s_j} (\sigma_j^{s'_j})^\dagger \right].
 \end{aligned} \tag{21}$$

After some algebra and using the above mapping $|k\rangle \otimes |k\rangle \rightarrow |k\rangle$ and $\frac{d}{d\Delta} = \frac{d\mu}{d\Delta} \frac{d}{d\mu} = -\frac{1}{\sqrt{1-\Delta^2}} \frac{d}{d\mu}$, we get

$$F_\Delta^{(0)} = \frac{\lambda^2 \mu^2}{2J^2(1-\Delta^2)} \langle L | \left(\sum_{k=1}^n T^{k-1} D T^{n-k} + \frac{1}{4} \frac{d^2 T^n}{d\eta^2} \right) | R \rangle, \tag{22}$$

with the vertex matrix

$$\begin{aligned}
 D &= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(\text{sgn}(1-\Delta^2) \frac{k^2}{2} |k\rangle \langle k| + \frac{k^2}{4} |k+1\rangle \langle k| \right. \\
 &\left. + \frac{(k+1)^2}{4} |k\rangle \langle k+1| \right).
 \end{aligned} \tag{23}$$

In analogy to $F_x^{(0)}$, the Fisher information $F_\Delta^{(0)}$ is also expressed in terms of the transfer matrix T with the insertion of the defect matrix D . Therefore, we can still use the combinatorial picture of paths going from $|R\rangle$ to $|L\rangle$, climbing and descending via intermediate states $\{|k\rangle\}_{k \geq 1}$, where one of the n th steps is ruled by the matrix element of D .

IV. SCALING OF THE FISHER INFORMATION AND THE RELATIVE ERROR

In this section, we study the scaling with the particle number n of the Fisher information and of the relative error of parameter estimations in different interaction regimes. A first general remark is that the validity condition (7) of the perturbation expansion (4), together with the lowest-order Fisher information (15), implies that the best relative error of $x \neq \Delta$ is pretty large for any value of $F_x^{(0)}$: $\frac{1}{x \sqrt{F_x^{(0)}}} > 1$. This is not the case for the anisotropy Δ , as we discuss in the following. Moreover, $F_x^{(0)}$ for any x show peculiar scalings in the three phases: isotropic interaction $|\Delta| = 1$, easy-plane interactions $|\Delta| < 1$, and easy-axis interactions $|\Delta| > 1$.

A. Isotropic limit $|\Delta| = 1$

We now show in the isotropic limit $\Delta \rightarrow 1$ and $\eta \rightarrow 0$ of the perturbative NESS how the computation of the low orders in η of the leading contribution for small $F_x^{(0)}$ is equivalent to the combinatorics of a random walk between states $|R\rangle$ and $|L\rangle$ with transition amplitudes given by the transfer matrix. First, we expand the sinusoidal functions in the transfer matrix (18) for small arguments $k\eta \ll 1 \forall k = 1, \dots, \lfloor \frac{n}{2} \rfloor$; thus, $\eta \ll \frac{2}{n}$: $T = T_0 + \sum_{l=1}^{\infty} \eta^{2l} T_{2l}$, with

$$T_0 = |L\rangle\langle L| + |R\rangle\langle R| + |1\rangle\langle 1| + \frac{|1\rangle\langle R| + |L\rangle\langle 1|}{2},$$

$$T_{2l>0} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left((-)^l \frac{(2k)^{2l}}{2(2l)!} |k\rangle\langle k| + (-)^{l-1} \frac{(2k)^{2l}}{4(2l)!} |k+1\rangle\langle k| \right. \\ \left. + (-)^{l-1} \frac{[2(k+1)]^{2l}}{4(2l)!} |k\rangle\langle k+1| \right). \quad (24)$$

Expanding the exponential T^n , the lowest orders in η of (17) and (22) are sums of matrix elements between states $|R\rangle$ and $|L\rangle$ of products with many transfer matrices T_0 and a few vertices $T_{2l>0}$ and D . Finally, we obtain

$$\langle L|T^n|R\rangle = \frac{n(n-1)}{8} - \frac{n(n-1)(n-2)}{24} \left(\eta^2 - \frac{\eta^4}{6}(3n-7) \right. \\ \left. + \frac{\eta^6}{180}[989 + 3n(36n-217)] \right) + O(\eta^8), \quad (25)$$

$$F_{\Delta}^{(0)} = \frac{\lambda^2}{96J^2} \mu^2 n(n-1)(n-2) \left(3n-7 \right. \\ \left. - \frac{\eta^2}{30}(n-3)(261n-799) \right) + O\left(\frac{\lambda^2}{J^2}\eta^4\right). \quad (26)$$

We already commented on the relative error for $x \neq \Delta$, which is larger than 1. The best relative error of Δ in the isotropic limit is

$$\frac{1}{\Delta F_{\Delta}^{(0)}} > O\left(\frac{1}{n^2}\right), \quad (27)$$

where we have used the validity condition (7) for small coupling $\frac{\lambda}{J}$, $\langle L|T^n|R\rangle = O(n^2)$ and $F_{\Delta}^{(0)} = O(n^4)$. Equation (27) implies that the relative error of the anisotropy Δ can decrease faster than the shot-noise limit with increasing size n , e.g., $1/n^{1+\alpha}$, with $\alpha \in (0, 1)$.

B. Anisotropic regime: Easy-plane interactions $|\Delta| < 1$

The scaling of Fisher information changes in the deep anisotropic regime. The leading orders in λ of $F_x^{(0)}$ for infinitely large n can be explicitly computed from the Jordan block of T corresponding to the largest eigenvalue: see Appendix A for the Jordan decomposition of the transfer matrix. We consider two complementary cases: a dense subinterval of $\Delta \in (-1, 1)$, namely, $\eta = \frac{q\pi}{p}$ with coprime integers p, q without loss of generality, and the case of irrational $\frac{\eta}{\pi}$. These cases exhibit different behaviors for $F_{\Delta}^{(0)}$.

In the first case $\eta = \frac{q\pi}{p}$, the transitions $\langle |p| \pm 1|T||p| \rangle$ vanish. Therefore, we can consider the restriction $T^{(d)}$ of

the transfer matrix (18) in the basis $\{|L\rangle, |R\rangle, |1\rangle, |2\rangle, \dots, |d\rangle\}$ with $d = |p| - 1$ for the computation of the matrix power in expression (17) of $F_x^{(0)}$ with $x \neq \Delta$. For irrational $\frac{\eta}{\pi}$, none of the transitions $\langle |p| \pm 1|T||p| \rangle$ vanish, and we need the full transfer matrix, with $d = \lfloor \frac{n}{2} \rfloor$. In both cases, the matrix $T^{(d)}$ is transformed into its Jordan canonical form via the similarity $T_j^{(d)} = (V^{(d)})^{-1} T^{(d)} V^{(d)}$; see Appendix A for details. The largest eigenvalue of $T^{(d)}$ is 1, corresponding to the eigenvector $|L\rangle$ and a defective eigenvector, and the other eigenvalues $\{\tau_j\}_{j=1, \dots, d}$ satisfy $1 > |\tau_1| \geq \dots \geq |\tau_d|$. One analytically computes

$$\langle L|T^n|R\rangle = \langle L|V^{(d)}(T_j^{(d)})^n(V^{(d)})^{-1}|R\rangle = \chi n + \chi_1, \quad (28)$$

with

$$\chi = \langle L|V^{(d)}|L\rangle\langle R|(V^{(d)})^{-1}|R\rangle = \frac{d}{2(d+1)} \frac{1}{1-\Delta^2},$$

$$\chi_1 = \langle L|(V^{(d)})^{-1}|R\rangle. \quad (29)$$

These equations show that the Fisher information $F_x^{(0)}$ with $x \neq \Delta$ is linear in the particle number. Figure 1 shows the coefficient χ for rational and irrational $\frac{\eta}{\pi}$ in the leading order for large n : note that there is no qualitative difference between the two cases. Remember, however, that despite the linear scaling of the Fisher information, the relative error of $x \neq \Delta$ is larger than 1 within the perturbation analysis (4) and (7).

The computation of $F_{\Delta}^{(0)}$ at the lowest order for $n \gg 1$ is more involved. The second term of (22) is

$$\frac{d^2}{d\eta^2} \langle L|T^n|R\rangle = \frac{d^2 \chi n}{d\eta^2} + O(1) = \frac{d}{d+1} \frac{2\Delta^2 + 1}{(1-\Delta^2)^2} n + O(1), \quad (30)$$

with $O(1)$ being constants in n . The other contribution of (22) differentiates between rational and irrational $\frac{\eta}{\pi}$. If $\eta = \frac{q\pi}{p}$, the transition $\langle |p| - 1|T||p| \rangle = 0$ implies that the transfer matrices on the left of the vertex matrix D can be restricted

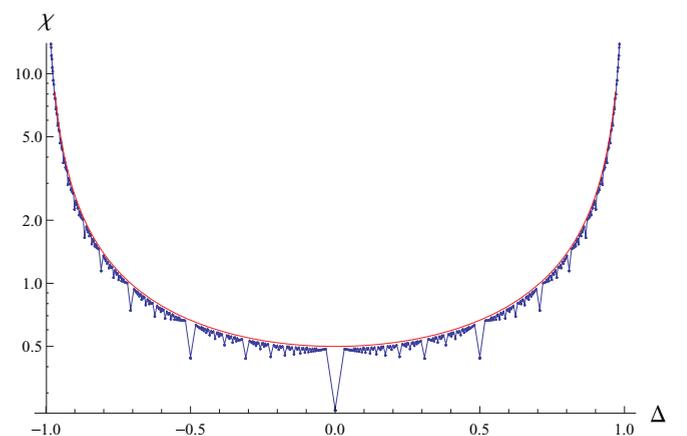


FIG. 1. (Color online) Semilog plots of the coefficients χ , determining the size scaling of Fisher information of NESS, as functions of Δ with $|\Delta| < 1$ for rational (dots joined by the blue line) $\frac{\eta}{\pi}$ and irrational (red line) $\frac{\eta}{\pi}$ for large n . The two lines perfectly overlap except for some spikes for small denominators of $\frac{\eta}{\pi}$.

to $T^{(d)}$ with $d = |p| - 1$ and so can the transfer matrices on the right of D except the one adjacent to D . The reason for this exception is that the transition $\langle |p| - 1 | D | |p| \rangle$ does not vanish, and thus, the T matrix to the right of D can lead the transition $\langle |p| - 1 \rangle \rightarrow \langle |p| \rangle$. We single out the latter contribution in Eq. (31). We again use the Jordan decomposition (A7) and $d = |p| - 1$:

$$\begin{aligned}
 \sum_{k=1}^n \langle L | T^{k-1} D T^{n-k} | R \rangle &= \sum_{k=1}^n \sum_{j=1}^d \sum_{l=1}^{d+1} \langle L | (T^{(d)})^{k-1} | j \rangle \langle j | D | l \rangle \langle l | (T^{(d)})^{n-k} | R \rangle \\
 &= \sum_{k=1}^n \sum_{j,l=1}^d \langle L | V^{(d)} (T_j^{(d)})^{k-1} (V^{(d)})^{-1} | j \rangle \langle j | D | l \rangle \langle l | V^{(d)} (T_j^{(d)})^{n-k} (V^{(d)})^{-1} | R \rangle \\
 &\quad + \sum_{k=1}^n \langle L | V^{(d)} (T_j^{(d)})^{k-1} (V^{(d)})^{-1} | d \rangle \langle d | D | d+1 \rangle \langle d+1 | T | d \rangle \langle d | V^{(d)} (T_j^{(d)})^{n-k-1} (V^{(d)})^{-1} | R \rangle \\
 &= \frac{d}{2(d+1)} \frac{n}{1-\Delta^2} \left(\sum_{j,l=1}^d \langle j | D | l \rangle \langle L | V^{-1} | j \rangle \bar{\psi}_l + \frac{(d+1)^2}{4} \frac{\sin(d\eta)}{2} \langle L | V^{-1} | d \rangle \bar{\psi}_d \right) \\
 &\quad + O(1) \equiv \xi_1 n + O(1). \tag{31}
 \end{aligned}$$

For irrational $\frac{\eta}{\pi}$, the transfer matrix in any position always equals $T^{(d)}$ with $d = \lfloor \frac{n}{2} \rfloor$. In this case, the first term of (22) becomes

$$\begin{aligned}
 \sum_{k=1}^n \langle L | T^{k-1} D T^{n-k} | R \rangle &= \sum_{k=1}^n \sum_{j,l=1}^d \langle L | (T^{(d)})^{k-1} | j \rangle \langle j | D | l \rangle \langle l | (T^{(d)})^{n-k} | R \rangle \\
 &= \sum_{k=1}^n \sum_{j,l=1}^d \langle L | V^{(d)} (T_j^{(d)})^{k-1} (V^{(d)})^{-1} | j \rangle \langle j | D | l \rangle \langle l | V^{(d)} (T_j^{(d)})^{n-k} (V^{(d)})^{-1} | R \rangle \\
 &= \frac{d}{2(d+1)} \frac{n}{1-\Delta^2} \sum_{j,l=1}^d \langle j | D | l \rangle \langle L | V^{-1} | j \rangle \bar{\psi}_l + O(1) \equiv \xi_1 n + O(1). \tag{32}
 \end{aligned}$$

Finally, we obtain at the leading order for large n

$$F_{\Delta}^{(0)} = \frac{\lambda^2 \mu^2}{J^2} \xi n + O(1), \quad \xi = \frac{1}{2(1-\Delta^2)} \left(\xi_1 - \frac{1}{4} \frac{d^2 \chi}{d\eta^2} \right). \tag{33}$$

The coefficient ξ is plotted in Fig. 2. It shows a very different behavior from χ , despite the fact that they are both nowhere continuous functions of Δ . While χ has a smooth envelope, ξ is constant in n but has a highly complex structure for rational $\frac{\eta}{\pi}$ with cusplike patterns which can be appreciated by consecutive zooms suggesting a fractal geometry (see Fig. 2). Moreover, ξ is unbounded for $|p| \rightarrow \infty$, indicating that the scaling of $F_{\Delta}^{(0)}$ is qualitatively different at irrational $\frac{\eta}{\pi}$. The differences between coefficients χ and ξ , and therefore between (17) and (22), are magnified for irrational $\frac{\eta}{\pi}$ since ξ depends on n , as shown in (32): ξn plotted in Fig. 3 exhibits piecewise power scaling $\propto n^\alpha$ with $\alpha \geq 2$ and an overall growth as fast as $\sim n^5$. Two remarks are in order. First, when $\frac{\eta}{\pi}$ is close to a rational number, such as $\Delta = \cos(\frac{\pi}{3}) - 10^{-4} = 0.4999$ and $\Delta = \cos(\frac{2\pi}{7}) + 10^{-4} \simeq 0.390117$ in Fig. 3, we observe the coefficient ξn grows like n^2 for a wide range of n and then eventually increases the slope; this is a signature of the transition toward rational values of $\frac{\eta}{\pi}$ where ξ is constant in n . Second, the oscillatory behavior in Fig. 3 is less pronounced when $\frac{\eta}{\pi}$ is an algebraic irrational number, such as the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ and $\sqrt{3}$ in Fig. 3.

The best relative error of Δ for easy-plane interactions is

$$\frac{1}{\Delta F_{\Delta}^{(0)}} > O\left(\frac{1}{\xi n}\right) = \begin{cases} O\left(\frac{1}{\xi}\right) & \text{for rational } \frac{\eta}{\pi}, \\ O\left(\frac{1}{n^4}\right) & \text{for irrational } \frac{\eta}{\pi}, \end{cases} \tag{34}$$

where we have used the validity condition (7) for small coupling $\frac{\lambda}{J}$, $\langle L | T^n | R \rangle = O(n)$ and $\xi n = O(n^5)$ for irrational $\frac{\eta}{\pi}$. Therefore, the relative error of the anisotropy Δ can be very small even though it does not decrease with n for rational $\frac{\eta}{\pi}$, while it can decrease faster than the shot-noise limit with increasing size n , e.g., $\sim 1/n^4$, for irrational $\frac{\eta}{\pi}$.

C. Anisotropic regime: Easy-axis interactions $|\Delta| > 1$

If $|\Delta| > 1$ ($\eta \in i\mathbb{R}$), $\langle L | T^n | R \rangle$ exhibits a superexponential scaling with n . This can be seen by noticing that $\langle L | T^n | R \rangle$ is the sum of positive terms, each representing the transition from $|R\rangle$ to $|L\rangle$ through different paths. For a very comfortable estimation we consider n to be even and only a single path with the first $\frac{n}{2}$ steps, which increase the index of the auxiliary basis, and the last $\frac{n}{2}$ steps, which decrease the index:

$$\langle L | A_{-}^{\frac{n}{2}} A_{+}^{\frac{n}{2}} | R \rangle = \frac{1}{2^n} \prod_{k=1}^{\frac{n}{2}-1} \sin^2(k\eta) \sin^2[(k+1)\eta]. \tag{35}$$

Using $\sinh^2 y \geq y^2 \forall y \in \mathbb{R}$, we find

$$\langle L | T^n | R \rangle > \frac{\eta^{2(n-2)}}{2^n} \left[\left(\frac{n}{2}\right)! \left(\frac{n}{2} - 1\right)! \right]^2, \tag{36}$$

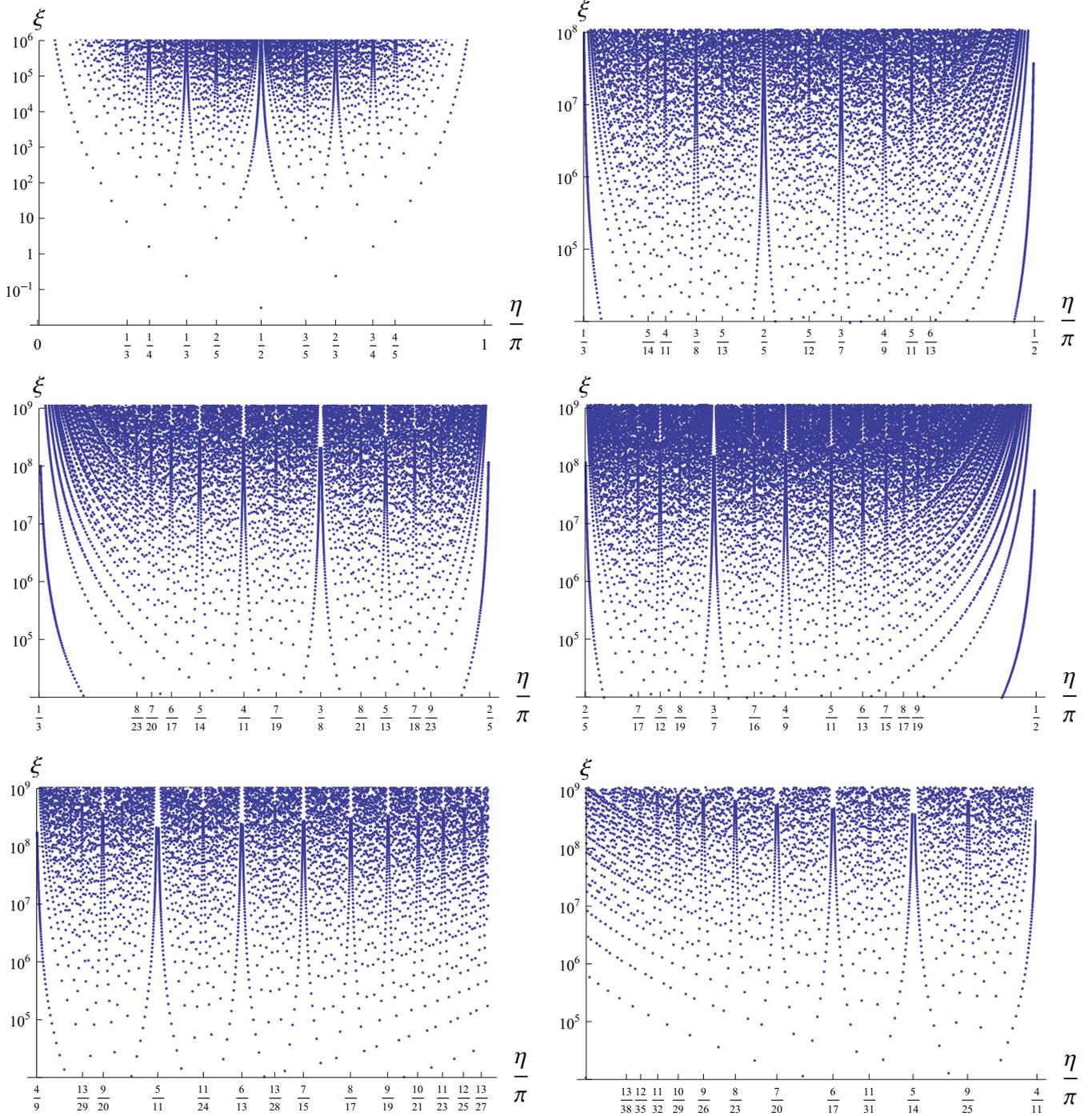


FIG. 2. (Color online) Semilog plots of the coefficient ξ of the Fisher information with respect to the anisotropy as a function of the anisotropy parameter $\frac{\eta}{\pi} \in [0, 1]$ for rational $\frac{\eta}{\pi}$. We notice that closer zooms reveal additional cusplike patterns, suggesting a fractal geometry. The blank spaces above some of the cusplike patterns are due to the numerical accuracy: we generated rational $\frac{\eta}{\pi} = \frac{q}{p}$ with $p \in [2, 1000]$ and $q \in [1, 1000]$, but for larger values of q and p the blank spaces are filled. For larger values of ξ , the less evident cusplike patterns become more evident.

which grows faster than any exponential of n . A similar reasoning holds for odd n and for $F_{\Delta}^{(0)}$.

Such superexponential scaling is observed only perturbatively in λ , where the approximation holds if higher orders in λ are small. From the nonperturbative solution at extreme driving $\mu = 1$ [56,60], with the NESS density operator (8), we have studied the behavior of F_{λ} on n numerically. We found that F_{λ}

grows superexponentially at small n and then quickly decays to zero for small coupling λ , even though, as we have shown before, the leading-order perturbative result keeps growing superexponentially. This can be understood as a crossover behavior from a highly correlated state when the correlation length $\ell = \ell(\Delta)$ is of the order of the system size $\ell \approx n$ to a very simple, macroscopically almost separable kink between

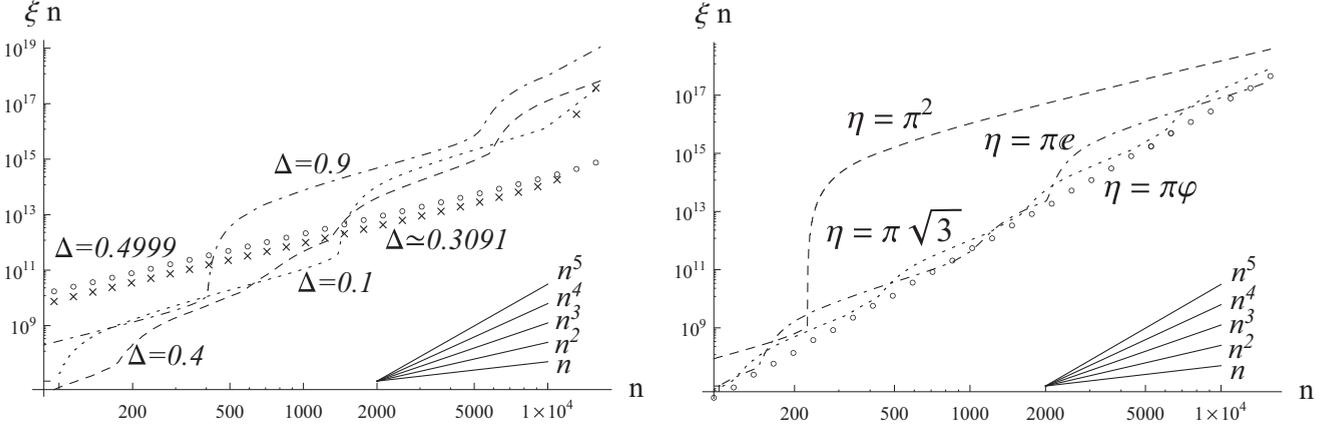


FIG. 3. Log-log plots of the coefficient ξn , which determines the size scaling of the Fisher information with respect to the anisotropy, as a function of n for irrational $\frac{\eta}{\pi}$. (left) $\Delta = 0.1$ (dotted line), $\Delta = 0.4$ (dashed line), $\Delta = 0.9$ (dot-dashed line), $\Delta = 0.4999$ (circles), and $\Delta = \cos(\frac{2}{5}\pi) + 10^{-4} \simeq 0.390117$ (crosses). (right) $\eta = \pi\varphi$ with $\varphi = \frac{1+\sqrt{5}}{2}$ being the golden ratio (circles), $\eta = \pi\sqrt{3}$ (dotted line), $\eta = \pi^2$ (dashed line), and $\eta = \pi e$ (dot-dashed line). For a comparison we also plotted the slopes of power laws n^1, n^2, n^3 , and n^4 (solid lines).

spin-up and spin-down ferromagnetic domains for $n \gg \ell$. In Fig. 4, we plot F_λ for $\Delta \in \{2, 10, 100\}$ and $\frac{\lambda}{J} \in \{0, 10^{-2}, 10^{-3}\}$, where the cases $\frac{\lambda}{J} = 0$ represent the leading orders of the perturbative expansions in $\frac{\lambda}{J}$.

We have already noted that the relative error of the estimation of λ is always pretty large, despite large values of the Fisher information F_λ . For F_Δ we only found small peaks, namely, $F_\Delta \lesssim 10$, and thus large relative error for the data in Fig. 4.

V. CONCLUSIONS

We have studied the NESS of an XXZ spin chain with local dissipation at its ends as a probe for the estimation of several

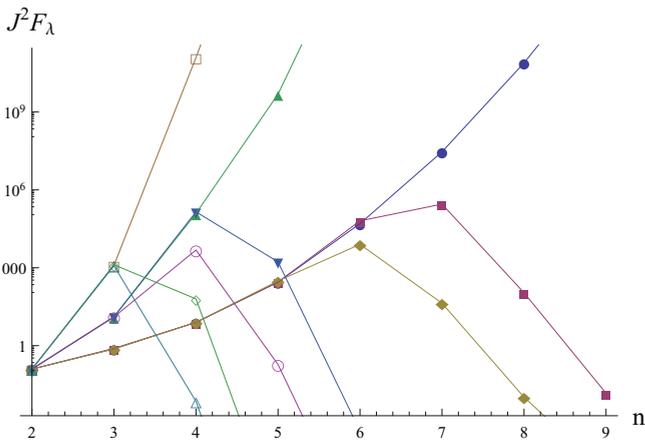


FIG. 4. (Color online) Semilog plot of $J^2 F_\lambda$ as a function of n : $(\Delta, \frac{\lambda}{J}) = (2, 0)$ (blue solid circles), $(\Delta, \frac{\lambda}{J}) = (2, 10^{-3})$ (purple solid squares), $(\Delta, \frac{\lambda}{J}) = (2, 10^{-2})$ (yellow solid diamonds), $(\Delta, \frac{\lambda}{J}) = (10, 0)$ (green solid upward triangles), $(\Delta, \frac{\lambda}{J}) = (10, 10^{-3})$ (blue downward triangles), $(\Delta, \frac{\lambda}{J}) = (10, 10^{-2})$ (purple open circles), $(\Delta, \frac{\lambda}{J}) = (10^2, 0)$ (yellow open squares), $(\Delta, \frac{\lambda}{J}) = (10^2, 10^{-3})$ (green open diamonds), and $(\Delta, \frac{\lambda}{J}) = (10^2, 10^{-2})$ (blue open upward triangles).

parameters, such as spin interaction strength J , anisotropy Δ , and dissipation parameters λ and μ . We quantified the efficiency of the above parameter estimations with the Fisher information and studied the scaling with the number of spins. We found superlinear scaling for small $\frac{\lambda}{J}$ close to the isotropic limit (XXX interaction) and in the easy-plane anisotropic phase ($|\Delta| < 1$) and superexponential scaling in the easy-axis anisotropic phase ($|\Delta| > 1$) within and beyond small perturbations in $\frac{\lambda}{J}$. We also considered the best relative error which can decrease with the particle number faster than shot noise only for the estimation of Δ in the isotropic limit and for easy-plane interactions with incommensurate $\frac{\eta}{\pi}$. This scaling identifies regimes for high-precision estimation of Δ , which is useful as a measure of bulk properties. Moreover, Δ can be interpreted as the phases of generalized nonlinear interferometers [19,21,23,32], where the phase to be estimated is proportional to the strength of the interparticle interaction in the chain.

The established enhanced metrological performances are not due to entanglement but are a different effect of many-body interactions out of equilibrium. Indeed, in the perturbative regime $\frac{\lambda}{J} \ll 1$, which is relevant for metrological purposes, the NESS is a perturbation of the completely mixed state and is thus unentangled since the completely mixed state is in the interior of the set of separable states [69].

Generalizations to more general Hamiltonians, such as XYZ interactions, or more general dissipation would be desirable. In the present framework, this requires the derivation of the corresponding NESS. Also, we stress that our results pertain to a completely integrable system with an exactly solvable NESS. Quantum metrology using NESS of generic, nonintegrable systems is an open problem left for future research.

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APPENDIX A: JORDAN DECOMPOSITION OF THE TRANSFER MATRIX

In this appendix we derive the Jordan decomposition of the transfer matrix restricted to the basis $\{|L\rangle, |R\rangle, |1\rangle, |2\rangle, \dots, |d\rangle\}$. First, let's rewrite the restricted transfer matrix as

$$T^{(d)} = |L\rangle\langle L| + |R\rangle\langle R| + \frac{|L\rangle\langle 1| + |1\rangle\langle R|}{2} + T^{(d)'}, \quad (\text{A1})$$

where $T^{(d)'}$ is the remainder of (18). The Jordan canonical form of $T^{(d)}$ is

$$T_J^{(d)} = (V^{(d)})^{-1} T^{(d)} V^{(d)} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \tau_1 & & \\ & & & \ddots & \\ & & & & \tau_d \end{pmatrix}, \quad (\text{A2})$$

where entries not shown are zeros. τ_j are both eigenvalues of $T^{(d)'}$ with eigenvectors $|\tau_j'\rangle$ and eigenvalues of $T^{(d)}$ with eigenvectors $|\tau_j\rangle = \frac{\langle 1|\tau_j'\rangle}{2\tau_j-2}|L\rangle + |\tau_j'\rangle$, arranged in decreasing order $|\tau_1| \geq \dots \geq |\tau_d|$. $T^{(d)}$ also has the eigenvalue 1 with eigenvector $|L\rangle$ and a defective eigenvector

$$\begin{aligned} |\psi\rangle &= \psi_R |R\rangle + \sum_{k=1}^d \psi_k |k\rangle, \\ \psi_R &= \frac{2(d+1)}{d} (1 - T_{1,1}^{(d)}) = \frac{2(d+1)}{d} (1 - \Delta^2), \quad (\text{A3}) \\ \psi_k &= \frac{2(d-k+1)}{d-k+2} \frac{T_{k,k-1}^{(d)}}{1 - T_{k,k}^{(d)}} \psi_{k-1}, \quad \psi_1 = 2 \end{aligned}$$

(see Appendix B). The matrix $V^{(d)}$ is defined by arranging the eigenvectors of $T^{(d)}$ in its columns:

$$V^{(d)} = \begin{pmatrix} 1 & 0 & \frac{\langle 1|\tau_1'\rangle}{2\tau_1-2} & \dots & \frac{\langle 1|\tau_d'\rangle}{2\tau_d-2} \\ 0 & \psi_R & 0 & \dots & 0 \\ 0 & \psi_1 & \left[\tau_1' \right] & \dots & \left[\tau_d' \right] \\ \vdots & \vdots & \left[\tau_1' \right] & \dots & \left[\tau_d' \right] \\ 0 & \psi_d & \left[\tau_1' \right] & \dots & \left[\tau_d' \right] \end{pmatrix}, \quad (\text{A4})$$

where we write the eigenvectors $|\tau_j'\rangle$ of $T^{(d)'}$ inside the square brackets.

If $\eta = \frac{q\pi}{p}$ with coprime integers p, q , the transitions $\langle |p| \pm 1 | T | |p| \rangle$ vanish, and we can use the restriction $T^{(d)}$ with $d = |p| - 1$ for the computation of the Fisher information. There is no loss of generality to consider coprime p, q since any rational $\frac{\eta}{\pi}$ can be reduced to the lowest terms. On the other hand, if $\frac{\eta}{\pi}$ is irrational, none of the transitions $\langle j \pm 1 | T | j \rangle$ vanish, and thus the full matrix $T^{(d)}$ with $d = \lfloor \frac{\eta}{2} \rfloor$ must be considered in the computations. In both cases, we can analyze the eigenvalues τ_j by means of the following matrix equality:

$$\begin{aligned} P^{-1}(\mathbb{1} - T^{(d)'})P &= \text{sgn}(1 - \Delta^2)DAD, \\ P &= |L\rangle\langle L| + |R\rangle\langle R| + \sum_{k=1}^d \left| \frac{\sin(\eta)}{\sin(k\eta)} \right| |k\rangle\langle k|, \end{aligned}$$

$$\begin{aligned} D &= \sum_{k=1}^d |\sin(k\eta)| |k\rangle\langle k|, \\ A &= \sum_{k=1}^d |k\rangle\langle k| - \frac{1}{2} \sum_{k=1}^{d-1} (|k\rangle\langle k+1| + |k+1\rangle\langle k|). \quad (\text{A5}) \end{aligned}$$

A is a Toeplitz tridiagonal matrix and thus has eigenvalues $1 - \cos(\frac{j\pi}{d+1})$ for $j = 1, \dots, d$ [70]. The positivity condition $A > 0$ implies

$$\begin{aligned} \mathbb{1} - T^{(d)'} > 0 &\Leftrightarrow |\tau_j| < 1 \text{ if } |\Delta| < 1, \\ \mathbb{1} - T^{(d)'} < 0 &\Leftrightarrow |\tau_j| > 1 \text{ if } |\Delta| > 1. \end{aligned} \quad (\text{A6})$$

Note that for $\eta = \frac{q\pi}{p}$ and $d \geq |p|$ the matrix P is singular, and indeed, the largest of the eigenvalues τ_j equals 1.

The Jordan canonical form of powers of the restricted matrix $T^{(d)}$ is used in the computation of the Fisher information:

$$(T_J^{(d)})^k = \begin{pmatrix} 1 & k & & & \\ 0 & 1 & & & \\ & & \tau_1^k & & \\ & & & \ddots & \\ & & & & \tau_d^k \end{pmatrix}. \quad (\text{A7})$$

APPENDIX B: COMPUTATION OF THE CONTINUOUS FRACTIONS C_k

The defective eigenvalue of the transfer matrix $T^{(d)}$ (18) is the solution of the equation

$$(T^{(d)} - \mathbb{1})|\psi\rangle = |L\rangle. \quad (\text{B1})$$

Using the knowledge of the entries of $T^{(d)}$, the latter equation can be recast in the following recurrence relation [60]:

$$\begin{aligned} |\psi\rangle &= \psi_R |R\rangle + \sum_{k=1}^d \psi_k |k\rangle, \quad \psi_R = 4C_{d-1}(1 - T_{1,1}^{(d)}), \\ \psi_k &= \frac{T_{k,k-1}^{(d)}\psi_{k-1}}{C_{d-k}(1 - T_{k,k}^{(d)})}, \quad \psi_1 = 2, \end{aligned} \quad (\text{B2})$$

with $C_0 = 1$ and $C_k = 1 - \frac{1}{4C_{k-1}}$. Note that the transfer matrix considered here is different from but similar to the one in [60], which is

$$\begin{aligned} P^{-1}TP, \\ P &= |L\rangle\langle L| + |R\rangle\langle R| + \sum_{k=1}^d \left| \frac{\sin(\eta)}{\sin(k\eta)} \right| |k\rangle\langle k|. \quad (\text{B3}) \end{aligned}$$

However, the components of the defective eigenvalues have the same recurrence structure [71].

The coefficients C_k are continued fractions of the form

$$C_k = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_k}}}}, \quad (\text{B4})$$

with $a_{2j} = 1$ and $a_{2j+1} = -4 \forall j \in \mathbb{N}$. See [72] for a reference on continued fractions. Equation (B4) is equal to

$$C_k = \frac{f_k}{g_k}, \quad (\text{B5})$$

with

$$\begin{aligned} f_k &= a_k f_{k-1} + f_{k-2}, & f_{-1} &= 1, & f_{-2} &= 0, \\ g_k &= a_k g_{k-1} + g_{k-2}, & g_{-1} &= 0, & g_{-2} &= 1. \end{aligned} \quad (\text{B6})$$

The first recursive equation in (B6) can be written as

$$\begin{aligned} \begin{pmatrix} f_{2k} \\ f_{2k+1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} f_{2k-2} \\ f_{2k-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix}^{n+1} \begin{pmatrix} f_{-2} \\ f_{-1} \end{pmatrix} \\ &= (-)^n \begin{pmatrix} 2k+1 & k+1 \\ -4k-4 & -2k-3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k+1 \\ -2k-3 \end{pmatrix}. \end{aligned} \quad (\text{B7})$$

Similarly, for the second recursive relation in (B6),

$$\begin{aligned} \begin{pmatrix} g_{2k} \\ g_{2k+1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} g_{2k-2} \\ g_{2k-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix}^{n+1} \begin{pmatrix} g_{-2} \\ g_{-1} \end{pmatrix} \\ &= (-)^n \begin{pmatrix} 2k+1 & k+1 \\ -4k-4 & -2k-3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2k+1 \\ -4k-4 \end{pmatrix}. \end{aligned} \quad (\text{B8})$$

From the ratio (B5), we compute $C_k = \frac{k+2}{2k+2}$. Thus, the components of $|\psi\rangle$ become

$$\begin{aligned} \psi_R &= \frac{2(d+1)}{d} (1 - T_{1,1}^{(d)}), \\ \psi_k &= \frac{2(d-k+1)}{d-k+2} \cdot \frac{T_{k,k-1}^{(d)}}{1 - T_{k,k}^{(d)}} \psi_{k-1}, \end{aligned} \quad (\text{B9})$$

as in Eq. (A3).

-
- [1] L. Spietz, K. W. Lehnert, I. Siddiqi, and R. J. Schoelkopf, *Science* **300**, 1929 (2003).
 - [2] D. M. Weld, P. Medley, H. Miyake, D. Hucul, D. E. Pritchard, and W. Ketterle, *Phys. Rev. Lett.* **103**, 245301 (2009).
 - [3] M. Brunelli, S. Olivares, and M. G. A. Paris, *Phys. Rev. A* **84**, 032105 (2011).
 - [4] Q. Zhou and T.-L. Ho, *Phys. Rev. Lett.* **106**, 225301 (2011).
 - [5] T. Müller, B. Zimmermann, J. Meineke, J.-P. Brantut, T. Esslinger, and H. Moritz, *Phys. Rev. Lett.* **105**, 040401 (2010).
 - [6] C. Sanner, E. J. Su, A. Keshet, R. Gommers, Y. I. Shin, W. Huang, and W. Ketterle, *Phys. Rev. Lett.* **105**, 040402 (2010).
 - [7] T. M. Stace, *Phys. Rev. A* **82**, 011611(R) (2010).
 - [8] A. E. Leanhardt, T. A. Pasquini, M. Saba, A. Schirotzek, Y. Shin, D. Kielpinski, D. E. Pritchard, and W. Ketterle, *Science* **301**, 1513 (2003).
 - [9] U. Marzolino and D. Braun, *Phys. Rev. A* **88**, 063609 (2013).
 - [10] W. Weng, J. D. Anstie, T. M. Stace, G. Campbell, F. N. Baynes, and A. N. Luiten, *Phys. Rev. Lett.* **112**, 160801 (2014).
 - [11] I. M. Savukov, S. J. Seltzer, M. V. Romalis, and K. L. Sauer, *Phys. Rev. Lett.* **95**, 063004 (2005).
 - [12] M. Loretz, T. Rosskopf, and C. L. Degen, *Phys. Rev. Lett.* **110**, 017602 (2013).
 - [13] K. Fang, V. M. Acosta, C. Santori, Z. Huang, K. M. Itoh, H. Watanabe, S. Shikata, and R. G. Beausoleil, *Phys. Rev. Lett.* **110**, 130802 (2013).
 - [14] C. D. Aiello, M. Hirose, and P. Cappellaro, *Nat. Commun.* **4**, 1419 (2013).
 - [15] Y. Eto, H. Ikeda, H. Suzuki, S. Hasegawa, Y. Tomiyama, S. Sekine, M. Sadgrove, and T. Hirano, *Phys. Rev. A* **88**, 031602(R) (2013).
 - [16] S. K. Steinke, S. Singh, P. Meystre, K. C. Schwab, and M. Vengalattore, *Phys. Rev. A* **88**, 063809 (2013).
 - [17] A. Waxman, Y. Schlussel, D. Groswasser, V. M. Acosta, L.-S. Bouchard, D. Budker, and R. Folman, *Phys. Rev. B* **89**, 054509 (2014).
 - [18] G. Puentes, G. Waldherr, O. Neumann, and J. Wrachtrup, [arXiv:1308.4349](https://arxiv.org/abs/1308.4349).
 - [19] A. Luis, *Phys. Lett. A* **329**, 8 (2004).
 - [20] B. L. Higgins, D. W. Berry, S. D. Bartlett, M. W. Mitchell, and G. J. Pryde, *Nature (London)* **450**, 393 (2007).
 - [21] S. Boixo, S. T. Flammia, C. M. Caves, and J. M. Geremia, *Phys. Rev. Lett.* **98**, 090401 (2007).
 - [22] S. Boixo, A. Datta, S. T. Flammia, A. Shaji, E. Bagan, and C. M. Caves, *Phys. Rev. A* **77**, 012317 (2008).
 - [23] S. M. Roy and S. L. Braunstein, *Phys. Rev. Lett.* **100**, 220501 (2008).
 - [24] B. L. Higgins, D. W. Berry, S. D. Bartlett, M. W. Mitchell, H. M. Wiseman, and G. J. Pryde, *New J. Phys.* **11**, 073023 (2009).
 - [25] L. Pezzé and A. Smerzi, *Phys. Rev. Lett.* **102**, 100401 (2009).
 - [26] T. Tilma, S. Hamaji, W. J. Munro, and K. Nemoto, *Phys. Rev. A* **81**, 022108 (2010).
 - [27] C. Gross, T. Zibold, E. Nicklas, J. Esteve, and M. K. Oberthaler, *Nature (London)* **464**, 1165 (2010).
 - [28] M. F. Riedel, P. Böhi, Y. Li, T. W. Hänsch, A. Sinatra, and P. P. Treutlein, *Nature (London)* **464**, 1170 (2010).
 - [29] F. Benatti, R. Floreanini, and U. Marzolino, *Ann. Phys. (NY)* **325**, 924 (2010).
 - [30] F. Benatti, R. Floreanini, and U. Marzolino, *J. Phys. B* **44**, 091001 (2011).
 - [31] G. Argentieri, F. Benatti, R. Floreanini, and U. Marzolino, *Int. J. Quantum Inf.* **9**, 1745 (2011).
 - [32] M. Napolitano, M. Koschorreck, B. Dubost, N. Behbood, R. J. Sewell, and M. W. Mitchell, *Nature (London)* **471**, 486 (2011).
 - [33] A. Rivas and A. Luis, *New J. Phys.* **14**, 093052 (2012).
 - [34] F. Wolfgramm, C. Vitelli, F. A. Beduini, N. Godbout, and M. W. Mitchell, *Nat. Photon.* **7**, 28 (2013).
 - [35] T. Berrada, R. Bücken, J.-F. Schaff, J. Schmiedmayer, T. Schumm, and S. van Frank, *Nat. Commun.* **4**, 2077 (2013).
 - [36] F. Benatti and D. Braun, *Phys. Rev. A* **87**, 012340 (2013).
 - [37] F. Benatti, R. Floreanini, and U. Marzolino, *Phys. Rev. A* **89**, 032326 (2014).
 - [38] C. Di Franco, M. Paternostro, and M. S. Kim, *Phys. Rev. Lett.* **102**, 187203 (2009).
 - [39] D. Burgarth and K. Maruyama, *New J. Phys.* **11**, 103019 (2009).

- [40] M. Wieśniak and M. Markiewicz, *Phys. Rev. A* **81**, 032340 (2010).
- [41] D. Burgarth, K. Maruyama, and F. Nori, *New J. Phys.* **13**, 013019 (2011).
- [42] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
- [43] F. Benatti and R. Floreanini, *Int. J. Mod. Phys. B* **19**, 3063 (2005).
- [44] J. Kolodynski and R. Demkowicz-Dobrzanski, *Phys. Rev. A* **82**, 053804 (2010).
- [45] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, *Nat. Phys.* **7**, 406 (2011).
- [46] S. F. Huelga, C. Macchiavello, T. Pellizzari, A. K. Ekert, M. B. Plenio, and J. I. Cirac, *Phys. Rev. Lett.* **79**, 3865 (1997).
- [47] Alex W. Chin, S. F. Huelga, and M. B. Plenio, *Phys. Rev. Lett.* **109**, 233601 (2012).
- [48] B. Bellomo, A. De Pasquale, G. Gualdi, and U. Marzolino, *Phys. Rev. A* **80**, 052108 (2009).
- [49] B. Bellomo, A. De Pasquale, G. Gualdi, and U. Marzolino, *J. Phys. A* **43**, 395303 (2010).
- [50] B. Bellomo, A. De Pasquale, G. Gualdi, and U. Marzolino, *Phys. Rev. A* **82**, 062104 (2010).
- [51] A. Monras and F. Illuminati, *Phys. Rev. A* **83**, 012315 (2011).
- [52] D. Braun and J. Martin, *Nat. Commun.* **2**, 223 (2011).
- [53] S. Alipour, M. Mehboudi, and A. T. Rezakhani, *Phys. Rev. Lett.* **112**, 120405 (2014).
- [54] E. G. Brown, W. Donnelly, A. Kempf, R. B. Mann, E. Martín-Martínez, and N. C. Menicucci, *New J. Phys.* **16**, 105020 (2014).
- [55] T. Prosen, *Phys. Rev. Lett.* **106**, 217206 (2011).
- [56] T. Prosen, *Phys. Rev. Lett.* **107**, 137201 (2011).
- [57] T. Prosen, *Phys. Scr.* **86**, 058511 (2012).
- [58] D. Karevski, V. Popkov, and G. M. Schutz, *Phys. Rev. Lett.* **110**, 047201 (2013).
- [59] T. Prosen, E. Ilievski, and V. Popkov, *New J. Phys.* **15**, 073051 (2013).
- [60] T. Prosen and E. Ilievski, *Phys. Rev. Lett.* **111**, 057203 (2013).
- [61] T. Prosen, *Nucl. Phys. B* **886**, 1177 (2014).
- [62] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1975).
- [63] A. S. Holevo, *Probabilistic and Statistical Aspect of Quantum Theory* (North-Holland, Amsterdam, 1982).
- [64] S. L. Braunstein, C. M. Caves, and G. J. Milburn, *Ann. Phys. (NY)* **247**, 135 (1996).
- [65] M. G. A. Paris, *Int. J. Quantum Inf.* **7**, 125 (2009).
- [66] H. Wichterich, M. J. Henrich, H.-P. Breuer, J. Gemmer, and M. Michel, *Phys. Rev. E* **76**, 031115 (2007).
- [67] N. P. Oxtoby, A. Rivas, S. F. Huelga, and R. Fazio, *New J. Phys.* **11**, 063028 (2009).
- [68] Given an eigenvalue z of the observable ζ , corresponding to the eigenvector $|z\rangle$, a measure of ζ with a system in the state ρ_∞ provides the value z with probability $\langle z|\rho_\infty|z\rangle$.
- [69] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States* (Cambridge University Press, Cambridge, 2006).
- [70] D. Kulkarni, D. Schmidt, and S.-K. Tsui, *Linear Algebra Appl.* **297**, 63 (1999).
- [71] We corrected the wrong formula of ψ_R written in [60]. However, the results of [60] still hold true because whenever ψ_R was evaluated, the right formula was actually used.
- [72] W. B. Jones and W. J. Thron, *Continued Fractions: Analytic Theory and Applications* (Cambridge University Press, Cambridge, 1980).