Riemann-Hilbert problems and integrable nonlinear partial differential equations, V

Dmitry Shepelsky

B. Verkin Institute for Low Temperature Physics, Kharkiv, Ukraine
and
V.N.Karazin Kharkiv National University

Step-like Cauchy problem for NLS

We consider the Cauchy problem for the focusing nonlinear Schrödinger (NLS) equation

$$\begin{aligned} &\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0, \qquad x \in \mathbb{R}, \quad t \geq 0, \\ &q(x,0) = q_0(x), \qquad x \in \mathbb{R}, \end{aligned}$$

where the initial data are assumed to approach, for lagre $|x|$, the non-zero backgrounds:

$$q_0(x) \sim \begin{cases} A_1 \mathrm{e}^{\mathrm{i}\phi_1} \mathrm{e}^{-2\mathrm{i}B_1 x}, & x \to -\infty \\ A_2 \mathrm{e}^{\mathrm{i}\phi_2} \mathrm{e}^{-2\mathrm{i}B_2 x}, & x \to +\infty, \end{cases}$$

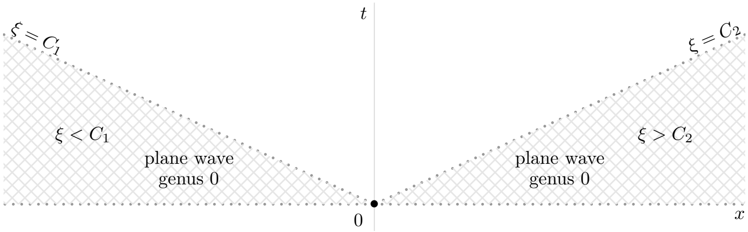
where $\{A_j, B_j, \phi_j\}_1^2$ are real constants; $A_j > 0$. The solution $q(x,t)$ is assumed to approach the associated plane wave backgrounds for all $t \geq 0$ ("nontrivial boundary conditions")

$$q(x,t) = q_{0j}(x,t) + \mathrm{o}(1) \ \text{ as } x \to (-1)^j \infty \text{ for all } t,$$

where

$$q_{0j}(x,t) = A_j \mathrm{e}^{\mathrm{i}\phi_1} \mathrm{e}^{-2\mathrm{i}B_j x + 2i\omega_j t} \quad \text{with} \ \ \omega_j = A_j^2 - 2B_j^2, \quad j = 1, 2.$$

Asymptotics in the case $B_1 = B_2$, $A_1 = A_2$



- $|\xi| > C(A, B)$ (here $C_1 = C_2 = C$): modulated plane waves

$$q(x,t) = A\mathrm{e}^{2\mathrm{i}(\omega t - Bx - \phi(\xi))} + O(t^{-1/2})$$

- $|\xi| < C$: modulated elliptic wave

$$q(x,t) = \hat{A}\frac{\Theta(\beta t + \gamma)}{\Theta(\beta t + \tilde{\gamma})}\mathrm{e}^{2\mathrm{i}(\nu t - \phi)} + O(t^{-1/2}).$$

Here $\hat{A}$, β, γ, $\tilde{\gamma}$, ν, ϕ are functions of $\xi = \frac{x}{t}$;
$\Theta(z) = \sum_{m\in\mathbb{Z}} \mathrm{e}^{\pi\mathrm{i}\tau m^2 + 2\pi\mathrm{i}mz}$ is the theta function of invariant $\tau(\xi)$.

Background solutions and Jost solutions

In what follows we are dealing with the case $B_1 \neq B_2$, $A_j \neq 0$, $j = 1, 2$.

- Determine solutions of Lax pair associated with background solutions of NLS $q_{0j}(x,t) = A_j \mathrm{e}^{-2\mathrm{i}B_j x + 2i\omega_j t + i\phi_j}$:

$$\Phi_{0j}(x,t,k) = \mathrm{e}^{(-\mathrm{i}B_j x + \mathrm{i}\omega_j t)\sigma_3} \mathcal{N}_{0j}(k) \mathrm{e}^{(-\mathrm{i}X_j(k)x - \mathrm{i}\Omega_j(k)t)\sigma_3}$$

where $X_j(k) = \sqrt{(k - E_j)(k - \bar{E}_j)}$, $E_j = B_j + iA_j$,
$\Omega_j(k) = 2(k + B_j)X_j(k)$,

$$\mathcal{E}_j(k) = \frac{1}{2}\begin{pmatrix} \varkappa_j(k) + \varkappa_j^{-1}(k) & \varkappa_j(k) - \varkappa_j^{-1}(k) \\ \varkappa_j(k) - \varkappa_j^{-1}(k) & \varkappa_j(k) + \varkappa_j^{-1}(k) \end{pmatrix}$$

with $\varkappa_j(k) = \left(\frac{k - E_j}{k - \bar{E}_j}\right)^{1/4}$, $\mathcal{N}_{0j}(k) = e^{\frac{i\phi_j}{2}} \mathcal{E}_j(k) e^{-\frac{i\phi_j}{2}}$.

- Let $q(x,t) \to q_{0j}(x,t)$ as $x \to (-1)^j \infty$. Then Jost solutions $\Phi_j(x,t,k)$ of Lax pair are fixed by ini. conds.:

$$\Phi_j \sim \Phi_{0j}, x \to (-1)^j \infty, \qquad k \in \mathbb{R} \cup (E_j, \bar{E}_j)$$

- scattering: $\Phi_2(x,t,k) = \Phi_1(x,t,k)s(k)$, $k \in \mathbb{R}$;
$s(k) = \begin{pmatrix} a^*(k) & b(k) \\ -b^*(k) & a(k) \end{pmatrix}$, where $f^*(k) := \overline{f(\bar{k})}$.

NLS with non-zero background: RH problem

Let $\phi_2 = 0$, $\phi_1 \equiv \phi$; $\Sigma_j := (E_j, \bar{E}_j)$. Define

$$M(x,t,k) := \begin{cases} \begin{pmatrix} \frac{\Phi_1^{(1)}}{a(k)} & \Phi_2^{(2)} \end{pmatrix} e^{(ikx+2ik^2t)\sigma_3}, & k \in \mathbb{C}^+, \\ \begin{pmatrix} \Phi_2^{(1)} & \frac{\Phi_1^{(2)}}{\bar{a}(\bar{k})} \end{pmatrix} e^{(ikx+2ik^2t)\sigma_3}, & k \in \mathbb{C}^-. \end{cases}$$

Then M satisfies the RH problem:

- $M_+(x,t,k) = M_-(x,t,k)J(x,t,k), \quad k \in \Sigma = \mathbb{R} \cup \Sigma_1 \cup \Sigma_2,$
- $M(x,t,\infty) = I,$

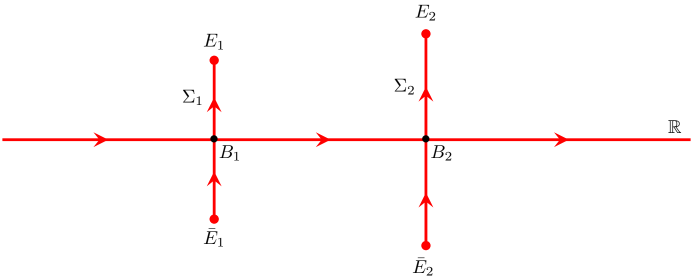
where $J(x,t,k) = e^{-(ikx+2ik^2t)\sigma_3} J_0(k) e^{(ikx+2ik^2t)\sigma_3}$ with $J_0(k)$ defined by

1. for $k \in \mathbb{R}$,

$$J_0(k) = \begin{pmatrix} 1 + r(k)r^*(k) & r^*(k) \\ r(k) & 1 \end{pmatrix} \qquad \text{with} \quad r(k) := \frac{b^*(k)}{a(k)};$$

2. for $k \in \Sigma_1 \cup \Sigma_2$,

$$J_0(k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{ie^{-i\phi}}{a_+ a_-} & 1 \end{pmatrix}, & k \in \Sigma_1 \cap \mathbb{C}^+ \\ \begin{pmatrix} \frac{a_-}{a_+} & i \\ 0 & \frac{a_+}{a_-} \end{pmatrix}, & k \in \Sigma_2 \cap \mathbb{C}^+ \end{cases} \qquad J_0(k) = \begin{cases} \begin{pmatrix} 1 & \frac{ie^{i\phi}}{a_+^* a_-^*} \\ 0 & 1 \end{pmatrix}, & k \in \Sigma_1 \cap \mathbb{C}^- \\ \begin{pmatrix} \frac{a_+^*}{a_-^*} & 0 \\ i & \frac{a_-^*}{a_+^*} \end{pmatrix}, & k \in \Sigma_2 \cap \mathbb{C}^- \end{cases}$$

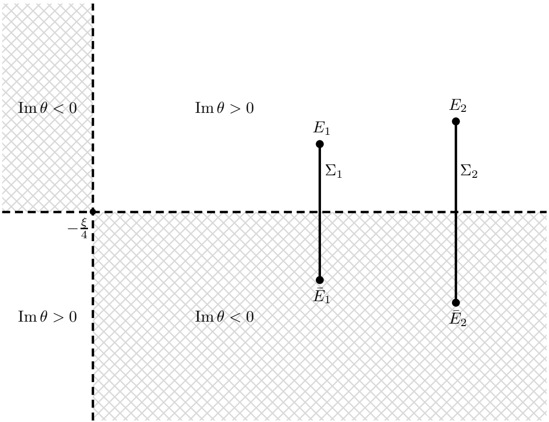


- Dependence on x and t of jump matrix: same as in the case of decaying ini. conditions, through $e^{(ikx+2ik^2t)\sigma_3} \equiv e^{it\theta(\xi,k)\sigma_3}$, where

$$\theta(\xi, k) = 2k^2 + \xi k \qquad (\xi = x/t).$$

- For large-t analysis, it would be nice to have that as $t \to \infty$, $J(x,t,k) \to \tilde{J}$ (piecewise) independent of k. Then the limiting RH problem can (hopefully) be solved explicitly, thus giving explicit asymptotics for $q(x,t)$.

RHP: dependence on x and t, II



Considering original RHP, we faces PROBLEM: at some parts of contour, depending on value of $\operatorname{Im}\theta(\xi, k)$,

$$e^{it\theta(\xi,k)} \text{ or } e^{-it\theta(\xi,k)} \text{ grows as } t \to \infty!$$

- SOLUTION: deform the contour and replace (in the jump matrix) the original "phase function" $\theta(\xi, k)$ by another phase function $g(\xi, k)$ ("g-function mechanism"), which has appropriate behavior on the (deformed) contour. $M \mapsto M^{(1)}$:

$$M^{(1)}(x,t,k) := \mathrm{e}^{-\mathrm{i}tg^{(0)}(\xi)\sigma_3} M(x,t,k)\mathrm{e}^{\mathrm{i}t(g(\xi;k)-\theta(\xi;k))\sigma_3}.$$

  - in order to keep large-k asymptotics for RHP:

$$g(\xi;k) = 2k^2 + \xi k + g^{(0)}(\xi) + \mathrm{O}(k^{-1})$$

  - deformed contour: $\hat{\Sigma} = \{k : \operatorname{Im} g(k) = 0\}$.

- RESULT: (i) appropriate $g(\xi;k)$ turn to be structurally different for different ranges of ξ; (ii) asymptotic structure depends on relationship amongst A_j, B_j.

“g-function mechanism”, II

- for $|\xi| >> 1$, the appropriate g-functions are (not surprising!) those involved in the construction of background solutions :

$$g(\xi;k) := \begin{cases} \Omega_1(k) + \xi X_1(k), & \xi << -1, \\ \Omega_2(k) + \xi X_2(k), & \xi >> 1, \end{cases}$$

where $X_j(k) = \sqrt{(k-E_j)(k-\bar{E}_j)}$, $\Omega_j(k) = 2(k+B_j)X_j(k)$

- in terms of derivative w.r.t. k: for $\xi >> 1$,

$$g'(\xi;k) = 4\frac{(k-\mu_1(\xi))(k-\mu_2(\xi))}{\sqrt{(k-E_2)(k-\bar{E}_2)}}$$

with $\mu_1(\xi) < \mu_2(\xi)$ real (similarly for $\xi << -1$, with E_2 replaced by E_1).

- Series of deformations of RH problem with this phase function lead finally to two model RH problems (applicable for $(-1)^j\xi >> 1$), each with a single jump arc; for $\xi >> 1$:

$$M_+^{(mod)}(k) = M_-^{(mod)}(k)\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ k \in \hat{\Sigma}_2$$

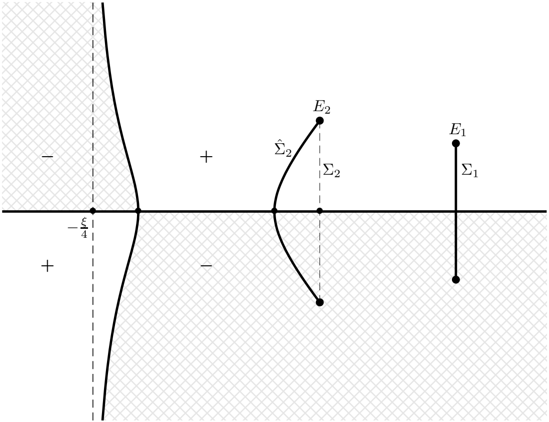
- model RHPs have explicit solutions: $M^{(mod)}(k) = \mathcal{E}_j(k)$, leading to

$$q(x,t) = A_j \mathrm{e}^{-2\mathrm{i}B_j x + 2\mathrm{i}\omega_j t - 2\mathrm{i}\psi(\xi)} + \mathrm{O}(t^{-\frac{1}{2}}), \qquad (-1)^j\xi >> 1,$$

with $\psi(\xi)$ determined by A_j, B_j, with $\psi(-\infty) = \phi_1$ and $\psi(+\infty) = \phi_2$.

Signature table

"Signature table" (distribution of signs of $\operatorname{Im} g(\xi, k)$) for $\xi >> 1$:



Here the jump matrix on Σ_1 decays exponentially fast to I as $t \to \infty$ and thus gives negligible contribution to the large-t asymptotics.

- Introducing g-function. $M \mapsto M^{(1)}$:
  $M^{(1)}(x,t,k) := \mathrm{e}^{-\mathrm{i}tg^{(0)}(\xi)\sigma_3} M(x,t,k)\mathrm{e}^{\mathrm{i}t(g(\xi;k)-\theta(\xi;k))\sigma_3}$.

$$M_+^{(1)}(x,t,k) = M_-^{(1)}(x,t,k)J^{(1)}(x,t,k), \quad k \in \mathbb{R} \cup \Sigma_1 \cup \hat{\Sigma}_2,$$

where

- $J^{(1)}(x,t,k) = \begin{pmatrix} 1+r(k)r^*(k) & r^*(k)e^{-2itg(\xi,k)} \\ r(k)e^{2itg(\xi,k)} & 1 \end{pmatrix}$, $k \in \mathbb{R}$,
- $J^{(1)}(x,t,k) = \begin{pmatrix} \frac{a_-(k)}{a_+(k)}e^{it(g_+-g_-)} & i \\ 0 & \frac{a_+(k)}{a_-(k)}e^{-it(g_+-g_-)} \end{pmatrix}$, $k \in \hat{\Sigma}_2 \cap \mathbb{C}^+$
  (and by symmetry for $\hat{\Sigma}_2 \cap \mathbb{C}^-$). Here we have used (for matrix entry (12)) that $g_+ + g_- = 0$ on $\hat{\Sigma}_2$.
  Since $\operatorname{Im} g_\pm = 0$ on $\hat{\Sigma}_2$, we have replaced the growth by oscillations (w.r.t t)!
- $J^{(1)}(x,t,k) = \begin{pmatrix} 1 & 0 \\ \frac{ie^{-i\phi}}{a_+(k)a_-(k)}e^{2ig(\xi,k)} & 1 \end{pmatrix}$, $k \in \Sigma_1 \cap \mathbb{C}^+$
  (and by symmetry for $\Sigma_1 \cap \mathbb{C}^-$).

Since $\operatorname{Im} g(\xi,k)=0$ for $k\in\mathbb{R}$, we "do lenses" along $k\in\mathbb{R}$ as in the decaying case but with $-\xi$ replaced by μ_1.

- Preparation for lenses. $M^{(1)}\mapsto M^{(2)}$:

$$M^{(2)}(x,t,k):=M^{(1)}(x,t,k)\delta^{-\sigma_3}(\xi,k)$$

with $\delta(\xi;k)=\exp\left\{\frac{1}{2\pi i}\int\limits_{-\infty}^{\mu_1(\xi)}\frac{\log(1-\lambda|r(s)|^2)}{s-k}ds\right\}$,

$$M_+^{(2)}(x,t,k)=M_-^{(2)}(x,t,k)J^{(2)}(x,t,k),\quad k\in\mathbb{R}\cup\hat{\Sigma}_2,$$

- $$J^{(2)}=\begin{cases}\begin{pmatrix}1 & \bar r\delta^2e^{-2itg}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\lambda r\delta^{-2}e^{2itg} & 1\end{pmatrix}, & k\in(\mu_1(\xi),\infty)\\ \begin{pmatrix}1 & 0\\ -\frac{\lambda r\delta_-^{-2}e^{2itg}}{1-\lambda|r|^2} & 1\end{pmatrix}\begin{pmatrix}1 & \frac{\bar r\delta_+^2e^{-2itg}}{1-\lambda|r|^2}\\ 0 & 1\end{pmatrix}, & k\in(-\infty,\mu_1(\xi))\end{cases}$$
- $$J^{(2)}(x,t,k)=\begin{pmatrix}\frac{a_-(k)}{a_+(k)}e^{2itg_+(\xi,k)} & i\delta^2(\xi,k)\\ 0 & \frac{a_+(k)}{a_-(k)}e^{2itg_-(\xi,k)}\end{pmatrix},\ k\in\hat{\Sigma}_2\cap\mathbb{C}^+$$

  (and by symmetry for $\hat{\Sigma}_2\cap\mathbb{C}^-$),

RHP deformations, III

Notice we have also oscillating jumps across $\hat{\Sigma}_2$, which means that we need lenses near $\hat{\Sigma}_2$ as well, suggested by the algebraic factorization (for $\hat{\Sigma}_2 \cap \mathbb{C}^+$)

$$\begin{pmatrix} e^{2itg_+(k)} & Y \\ 0 & e^{2itg_-(k)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Y^{-1}e^{2itg_-(k)} & 1 \end{pmatrix} \begin{pmatrix} 0 & Y \\ -Y^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Y^{-1}e^{2itg_+(k)} & 1 \end{pmatrix}$$

But in our case, the "lenses near $\mathbb{R}$" serve $\hat{\Sigma}_2$ as well.

- It would be nice to have $\operatorname{Im} g(k) > 0$ for all k near $\hat{\Sigma}_2 \cap \mathbb{C}^+$ (and, by symmetry, $\operatorname{Im} g(k) < 0$ for all k near $\hat{\Sigma}_2 \cap \mathbb{C}^-$).

- Doing lenses. $M^{(2)} \mapsto M^{(3)}$:

$$M^{(3)}(x,t,k) := M^{(2)}(x,t,k) \begin{cases} \begin{pmatrix} 1 & 0 \\ \lambda r(k)\delta^{-2}(k)e^{2itg(\xi,k)} & 1 \end{pmatrix}, & k \in \hat{\Omega}_2 \\ \begin{pmatrix} 1 & \bar{r}(k)\delta^{2}(k)e^{-2itg(\xi,k)} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\Omega}_3 \\ \begin{pmatrix} 1 & -\frac{\bar{r}(k)\delta_+^2(k)e^{-2itg(\xi,k)}}{1-\lambda|r|^2} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\Omega}_1 \\ \begin{pmatrix} 1 & 0 \\ -\frac{\lambda r(k)\delta_-^{-2}(k)e^{2itg(\xi,k)}}{1-\lambda|r|^2} & 1 \end{pmatrix}, & k \in \hat{\Omega}_4 \end{cases}$$

$$M_+^{(3)}(x,t,k) = M_-^{(3)}(x,t,k)J^{(3)}(x,t,k), \quad k \in \cup_{j=1}^4 \hat{\gamma}_j \cup \hat{\Sigma}_2,$$

- $J^{(3)}$ across $\hat{\gamma}_j$, $j = 1, \ldots, 4$ are triangular matrices as above
- important: $\hat{\Sigma}_2 \in \hat{\Omega}_2 \cup \hat{\Omega}_3$; then all these triangular jumps decay to I as $t \to \infty$
- $J^{(3)} = J^{(3)}(\xi,k) = \begin{pmatrix} 0 & i\delta^2(\xi,k) \\ i\delta^{-2}(\xi,k) & 0 \end{pmatrix}$, $k \in \hat{\Sigma}_2$ **(doesn't involve large parameter!)**

RHP deformations, V

- Getting rid of k dependence of jump. $M^{(3)} \mapsto M^{(mod)}$:

$$M^{(mod)}(x,t,k) := \Delta^{\sigma_3}(\xi,\infty)M^{(3)}(x,t,k)\Delta^{-\sigma_3}(\xi,k),$$

where Δ solves scalar RHP: find $\Delta(\xi,k)$ analytic in $\mathbb{C}\setminus\hat{\Sigma}_2$ and bounded as $k\to\infty$ satisfying "jump" cond.

$$\Delta_+(\xi,k)\Delta_-(\xi,k) = \delta^{-2}(\xi,k), \quad k\in\hat{\Sigma}_2.$$

Then
$$M_+^{(mod)}(x,t,k) = M_-^{(mod)}(x,t,k)\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad k\in\hat{\Sigma}_2.$$

  - RHP for Δ reduces to: $\log\Delta_+ + \log\Delta_- = \log\delta^{-2}$, or $\left(\frac{\log\Delta}{X}\right)_+ - \left(\frac{\log\Delta}{X}\right)_- = \frac{\log\delta^{-2}}{X_+}$, where $X(k) = \sqrt{(k-E_2)(k-\bar{E}_2)}$.
  - Solution by Cauchy int. $\Delta(\xi,k) = \exp\left\{\frac{X(k)}{2\pi i}\int_{\hat{\Sigma}_2}\frac{\log\delta^{-2}(\xi,s)ds}{X_+(s)(s-k)}\right\}$
  - RHP for $M^{(mod)}$ is solved explicitly.

- Following back the transformations $M^{(mod)} \mapsto M^{(3)} \mapsto M^{(2)} \mapsto M^{(1)} \mapsto M \mapsto q$ we obtain

$$q(x,t) = A_j \mathrm{e}^{-2\mathrm{i}B_j x + 2\mathrm{i}\omega_j t - 2\mathrm{i}\psi(\xi)} + \mathrm{O}(t^{-\frac{1}{2}}), \qquad (-1)^j\xi >> 1.$$

Why g-mechanism works

Properties of the g-function used in the deformations above:

1. $\operatorname{Im} g_\pm(k) = 0,\ k \in \hat{\Sigma}_2$.
2. $\operatorname{Im} g(k) > 0$ "near $\hat{\Sigma}_2 \cap \mathbb{C}^+$", $\operatorname{Im} g(k) < 0$ "near $\hat{\Sigma}_2 \cap \mathbb{C}^-$".
3. $\operatorname{Im} g(k) = 0,\ k \in \mathbb{R}$.
4. $g_+(k) + g_-(k) = 0,\ k \in \hat{\Sigma}_2$.
5. $\operatorname{Im} g(k) > 0,\ k \in \Sigma_1 \cap \mathbb{C}^+$ and $\operatorname{Im} g(k) < 0,\ k \in \Sigma_1 \cap \mathbb{C}^-$.
6. $g(\xi, k) = 2k^2 + \xi k + g^0(\xi) + \mathrm{O}(k^{-1})$

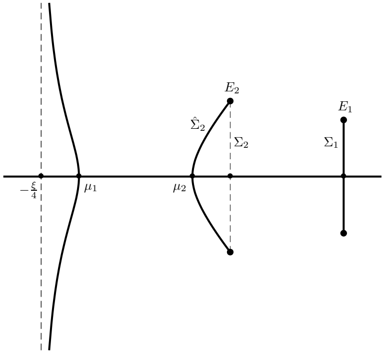
These properties determine $g(k)$ uniquely!

$B_2 < B_1$: rarefaction, I

Decreasing $|\xi|$, this type of g-function stops to work when

1. (i) either $\mu_1(\xi)$ and $\mu_2(\xi)$ collides,
2. (ii) or the infinite branch of $\{k : \operatorname{Im} g(k) = 0\}$ hits E_j.

If $B_2 < B_1$, it is always (i) that occurs:



$\mu_1(\xi)$ and $\mu_2(\xi)$ merge at $\xi = C_2 = -4B_2 + 4\sqrt{2}A_2$ and $\xi = C_1 = -4B_1 - 4\sqrt{2}A_2$, which signifies the ends of the plane wave sectors and the necessity of a new g-function.

$B_2 < B_1$: rarefaction, II

The transition to the new sectors is characterized by the emergence of two complex zeros of $g'(\xi;k)$, $\beta(\xi)$ and $\bar{\beta}(\xi)$, at the place of merging real zeros (keeping one real zero μ):

$$g'(\xi;k) = 4\frac{(k-\mu(\xi))(k-\beta(\xi))(k-\bar{\beta}(\xi))}{\sqrt{(k-E_2)(k-\bar{E}_2)(k-\beta(\xi))(k-\bar{\beta}(\xi))}}$$

for $-4B_2 < \xi < -4B_2 + 4\sqrt{2}A_2$; similarly for $-4B_1 - 4\sqrt{2}A_1 < \xi < -4B_1$.

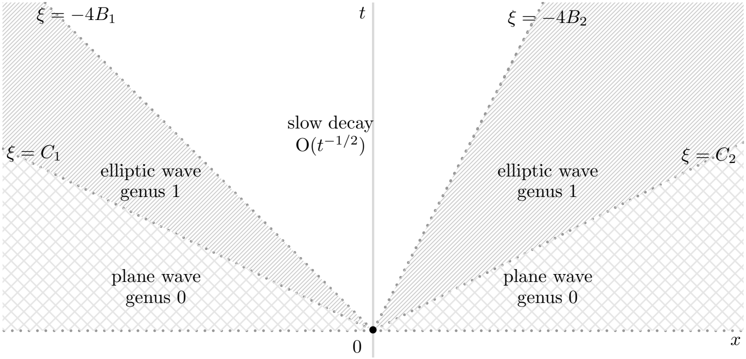
1. The associated model RHP has jumps across two arcs: $\Sigma = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$, $M_+^{(mod)}(x,t,k) = M_-^{(mod)}(x,t,k)\begin{pmatrix} 0 & i\mathrm{e}^{ixD_jx+itG_jt+\phi_j} \\ i\mathrm{e}^{-ixD_jx-itG_jt-\phi_j} & 0 \end{pmatrix}$, $k \in \tilde{\Sigma}_j$, $j=1,2$ (on each arc, the jump is independent of k!).
2. The solution can be given in terms of Riemann theta functions of dimension 1 (genus-1 solution):

$$q(x,t) = \hat{A}_j \frac{\Theta(\beta_j t + \gamma_j)}{\Theta(\beta_j t + \tilde{\gamma}_j)}\mathrm{e}^{\mathrm{i}\nu_j t} + O(t^{-1/2}),$$

where all coefficients are functions of $\xi = \frac{x}{t}$.

$B_2 < B_1$: rarefaction, III

For $-4B_1 < \xi < -4B_2$: the original phase function $it\theta(\xi;k)$ is such that the jumps in the original RH problem across both arcs Σ_1 and Σ_2 decay (exponentially) to the identity matrix as $t \to \infty$ and thus one can keep $g(\xi;k) = \theta(\xi;k)$ for this range. It follows that the asymptotic analysis in this sector essentially follows that in the case of the zero background, giving $q(x,t) = \mathrm{O}(t^{-1/2})$.



$B_2 > B_1$: shock, I

The case $B_2 > B_1$ turns out to be much richer: there are several asymptotic scenarios depending on the values $A_1/(B_2 - B_1)$ and $A_2/(B_2 - B_1)$, each being characterized by a set of appropriate g-functions; but there are always two infinite branches of $\operatorname{Im} g(\xi; k) = 0$: the real axis and a branch approaching the vertical line $\operatorname{Re} k = -\xi/4$.

In what follows, for simplicity we consider the symmetric case where $A_1 = A_2 = A$ and $B_2 = -B_1 = B > 0$. In this case, the asymptotic picture is symmetric in ξ and thus it is sufficient to consider ξ ranging from $+\infty$ down to $\xi = 0$.

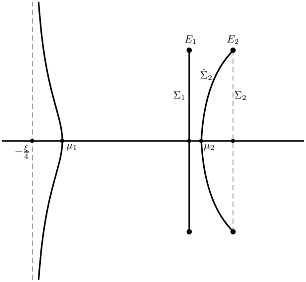
$B_2 > B_1$: shock, II

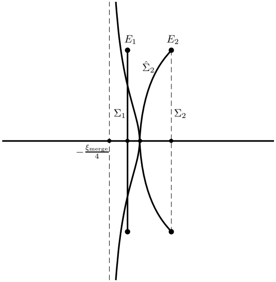
As ξ decreases from $+\infty$, there are three possibilities for ending the plane wave sector:

1. If $\frac{A}{B} > \frac{2}{7}(2+3\sqrt{2})$, then μ_1 and μ_2 merge, at $\xi = \xi^{(1)}_{merge} = -4B + 4\sqrt{2}A$, **before** the infinite branch hits E_1 and $\bar{E}_1$.
2. If $\frac{A}{B} = \frac{2}{7}(2+3\sqrt{2})$, then the infinite branch hits E_1 and $\bar{E}_1$ at the same moment $\xi = \frac{4}{7}(4\sqrt{2}+5)B$ as μ_1 and μ_2 merge.
3. If $\frac{A}{B} < \frac{2}{7}(2+3\sqrt{2})$, then the infinite branch hits E_1 and $\bar{E}_1$, at $\xi = \xi^{(1)}_{E_1} = 2(B + \sqrt{A^2+B^2})$, **before** μ_1 and μ_2 merge.

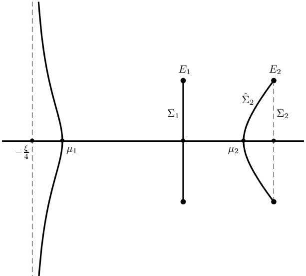
$B_2 > B_1$: shock, III

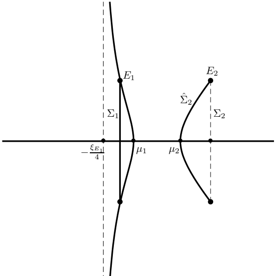
Case (i):





Case (iii):





$B_2 > B_1$, case (i): $\frac{A}{B} > \frac{2}{7}(2+3\sqrt{2})$. Four types of asymptotics, I

- the asymptotics in the range $\xi^{(2)}_{E_1} < \xi < \xi^{(1)}_{merge}$ is characterized, similarly to the rarefaction case, by the genus-1 g-function

$$g'(\xi;k) = 4\frac{(k-\mu(\xi))(k-\beta(\xi))(k-\bar{\beta}(\xi))}{\sqrt{(k-E_2)(k-\bar{E}_2)(k-\beta(\xi))(k-\bar{\beta}(\xi))}}$$

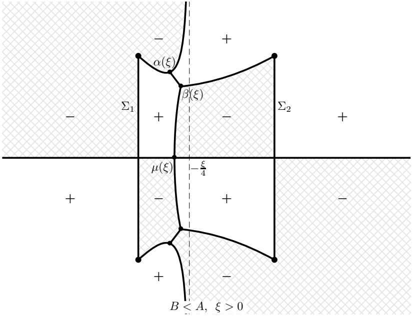
- The left end $\xi^{(2)}_{E_1}$ of the genus-1 range: when the infinite branch of g function above hits E_1 and $\bar{E}_1$: for $\xi < \xi^{(2)}_{E_1}$, a new, genus-3 g-function

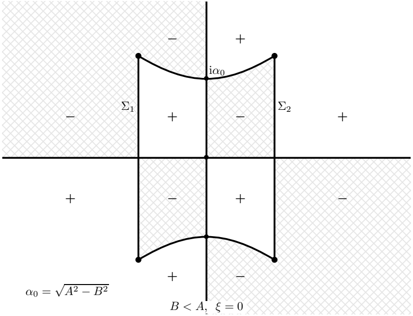
$$g'(\xi;k) = 4\frac{(k-\mu(\xi))(k-\beta(\xi))(k-\bar{\beta}(\xi))(k-\alpha(\xi))(k-\bar{\alpha}(\xi))}{\sqrt{(k-E_1)(k-\bar{E}_1)(k-E_2)(k-\bar{E}_2)(k-\beta(\xi))(k-\bar{\beta}(\xi))(k-\alpha(\xi))(k-\bar{\alpha}(\xi))}}$$

  becomes appropriate, with $\alpha(\xi)$ "emerging from E_1".
- The left end of the genus-3 range is $\xi = 0$: as $\xi \to 0$, $\alpha(\xi)$ and $\beta(\xi)$ both approach a single point $\mathrm{i}\alpha_0 = \mathrm{i}\sqrt{A^2 - B^2}$ whereas $\mu(\xi) \to 0$, and the g-function takes a genus-1 form:

$$g'(0;k) = 4\frac{k(k^2+\alpha_0^2)}{\sqrt{(k-E_1)(k-\bar{E}_1)(k-E_2)(k-\bar{E}_2)}}$$

$B_2 > B_1$, case (i): $\frac{A}{B} > \frac{2}{7}(2+3\sqrt{2})$. Four types of asymptotics, II





$$\frac{A}{B} > \frac{2}{7}(2+3\sqrt{2})$$

$\xi = 0$	$0 < \xi < \xi_{E_1}^{(2)}$	$\xi = \xi_{E_1}^{(2)}$	$\xi_{E_1}^{(2)} < \xi < \xi_{merge}^{(1)}$	$\xi = \xi_{merge}^{(1)}$	$\xi > \xi_{merge}^{(1)}$
genus 1	genus 3		genus 1		genus 0
α, β merge		the infinite branch hits E_1, $\bar{E}_1$		the real zeros μ_1, μ_2 merge	

RHP for genus-N solutions

- The model RHP associated with a genus-N solution has jumps across $N+1$ arcs: $\Sigma = \cup_{j=1}^{N+1}\tilde{\Sigma}_j$,
  $M_+^{(mod)}(x,t,k) =$
  $M_-^{(mod)}(x,t,k) \begin{pmatrix} 0 & ie^{ixD_jx+itG_jt+\phi_j} \\ ie^{-ixD_jx-itG_jt-\phi_j} & 0 \end{pmatrix}$, $k \in \tilde{\Sigma}_j$,
  $j = 1, \dots, N+1$ (on each arc, the jump is independent of k!); D_j, G_j are determined by the arc ends.
- The solution $M^{(mod)}$ (and, consequently, $q^{(ass)}(x,t)$) can be given in terms of Riemann theta functions of dimension N (genus-N solution):

$$q^{(ass)}(x,t) = \alpha \frac{\Theta(\mathbf{B}t + \mathbf{F})}{\Theta(\mathbf{B}t + \tilde{\mathbf{F}})} e^{i\nu t},$$

  $\mathbf{B}$ and $\mathbf{F}$ are N-component vectors; all coefficients are functions of ξ.
- The Riemann theta function $\Theta(u_1, \dots, u_N)$ associated with τ (matrix of periods) is defined for $\mathbf{u} \in \mathbb{C}^N$ by the Fourier series

$$\Theta(u_1, \dots, u_N) = \sum_{\mathbf{l} \in \mathbb{Z}^N} \exp\{\pi i(\tau \mathbf{l}, \mathbf{l}) + 2\pi i(\mathbf{l}, \mathbf{u})\},$$

  where $(\mathbf{l}, \mathbf{u}) = l_1 u_1 + \ldots + l_n u_N$.

This case, comparing to Case (i), is characterized by the equality $\xi_{E_1}^{(2)} = \xi_{merge}^{(1)}$; thus, the genus-3 range $0 < \xi < \xi_{merge}^{(1)}$ is directly adjacent to the plane wave range (no genus-1 range).

$$\frac{A}{B} = \frac{2}{7}(2+3\sqrt{2})$$

$\xi = 0$	$0 < \xi < \xi_{E_1}^{(2)}$	$\xi = \xi_{E_1}^{(2)} = \xi_{merge}^{(1)}$	$\xi > \xi_{merge}^{(1)}$
genus 1	genus 3		genus 0
α, β merge		the infinite branch hits E_1, $\bar{E}_1$ and the real zeros μ_1, μ_2 merge	

In this case, the infinite branch of the plane wave g-function hits E_1 and $\bar{E}_1$, leading, for ξ just to the left of $\xi_{E_1}^{(1)}$, to a genus-2 g-function with the complex zeros at $\alpha(\xi)$ and $\bar{\alpha}(\xi)$ ("emerging" from E_1 and $\bar{E}_1$):

$$g'(\xi;k) = 4\frac{(k-\mu_1(\xi))(k-\mu_2(\xi))(k-\alpha(\xi))(k-\bar{\alpha}(\xi))}{\sqrt{(k-E_1)(k-\bar{E}_1)(k-E_2)(k-\bar{E}_2))(k-\alpha(\xi))(k-\bar{\alpha}(\xi))}}$$

- There is the second "bifurcation value" of A/B: $A/B = 1$, separating, when ξ is decreasing, three different scenarios of appearance of further asymptotic ranges.

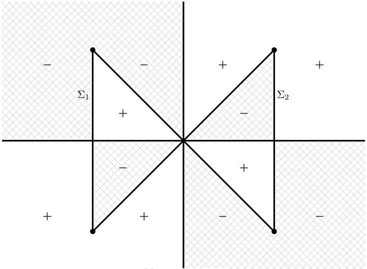
- Case (iii-a): $\frac{A}{B} < 1$. In this case, the left end $\xi_{merge}^{(2)}$ of the genus-2 sector corresponds to merging of $\alpha(\xi)$ and $\bar{\alpha}(\xi)$ into a third real zero, μ_0, leading to a genus-1 anzats (different from above!) for the g-function in $-\xi_{merge}^{(2)} < \xi < \xi_{merge}^{(2)}$ (thus including $\xi = 0$).

$$g'(\xi; k) = 4\frac{(k - \mu_1(\xi))(k - \mu_2(\xi))(k - \mu_0(\xi))}{\sqrt{(k - E_1)(k - \bar{E}_1)(k - E_2)(k - \bar{E}_2))}}$$

$$0 < \frac{A}{B} < 1$$

$0 \le \xi < \xi_{merge}^{(2)}$	$\xi = \xi_{merge}^{(2)}$	$\xi_{merge}^{(2)} < \xi < \xi_{E_1}$	$\xi = \xi_{E_1}$	$\xi > \xi_{E_1}$
genus 1		genus 2		genus 0
residual region	α, $\bar{\alpha}$ merge into a third real zero	transition region	infinite branch hits E_1, $\bar{E}_1$	wave plane region

- Case (iii-b): $\frac{A}{B} = 1$. In this case, $\xi^{(2)}_{merge}$ becomes 0 and thus the genus-1 range from Case (iii-a) shrinks to a single value $\xi = 0$, with $\alpha_0 = 0$.



(c) $B = A$, $\xi = 0$

$\frac{A}{B} = 1$

$\xi = 0$	$0 < \xi < \xi_{E_1}$	$\xi = \xi_{E_1}$	$\xi > \xi_{E_1}$
genus 1	genus 2		genus 0
α, $\bar{\alpha}$, μ_1 all merge at the origin		the infinite branch hits E_1, $\bar{E}_1$	

- Case (iii-c): $1 < \frac{A}{B} < \frac{2}{7}(2+3\sqrt{2})$. In this case, the left end $\xi_{merge}^{(2)}$ of the genus-2 sector $\xi_{merge}^{(2)} < \xi < \xi_{E_1}^{(1)}$ corresponds to merging of the real zeros $\mu_1(\xi)$ and $\mu_2(\xi)$ and emerging a pair of complex zeros $\beta(\xi)$ and $\bar{\beta}(\xi)$ thus leading to the genus-3 g-function for the range $0 < \xi < \xi_{merge}^{(2)}$.

$1 < \frac{A}{B} < \frac{2}{7}(2+3\sqrt{2})$

$\xi = 0$	$0 < \xi < \xi_{merge}^{(2)}$	$\xi = \xi_{merge}^{(2)}$	$\xi_{merge}^{(2)} < \xi < \xi_{E_1}^{(1)}$	$\xi = \xi_{E_1}^{(1)}$	$\xi > \xi_{E_1}^{(1)}$
genus 1	genus 3		genus 2		genus 0
α, β merge		the real zeros μ_1, μ_2 merge		the infinite branch hits E_1, $\bar{E}_1$	

Initial boundary value (IBV) problem for focusing NLS

Let $q(x,t)$ be the solution of the IBV problem for focusing NLS:

- $iq_t + q_{xx} + 2|q|^2 q = 0, \qquad x > 0, t > 0,$
- $q(x,0) = q_0(x)$ fast decaying as $x \to +\infty$
- $q(0,t) = g_0(t)$ time-periodic $\boxed{g_0(t) = \alpha\, \mathrm{e}^{2i\omega t}}$ $\quad \alpha > 0, \omega \in \mathbb{R}$
  $(q(0,t) - \alpha\, \mathrm{e}^{2i\omega t} \to 0$ as $t \to +\infty)$

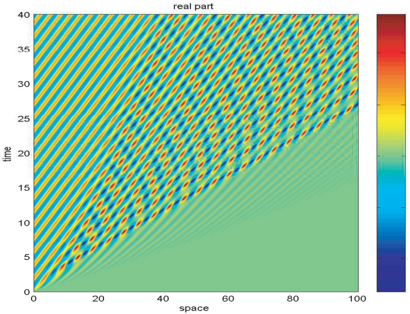
- Question: How does $q(x,t)$ behave for large t?
- Numerics: Qualitatively different pictures for parameter ranges:

  (i) $\omega < -3\alpha^2$

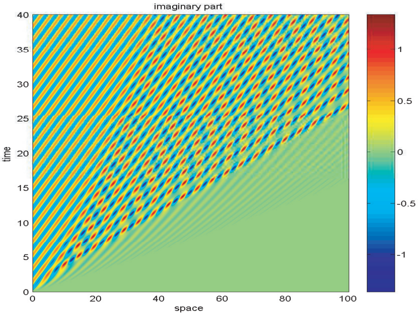
  (ii) $-3\alpha^2 < \omega < \frac{\alpha^2}{2}$

  (iii) $\omega > \frac{\alpha^2}{2}$

Numerics for $\omega < -3\alpha^2$, I



Real part $\mathrm{Re}\, q(x,t)$



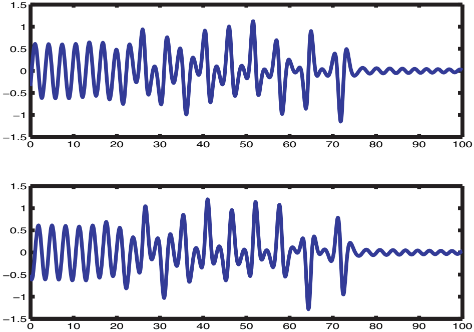
Imaginary part $\mathrm{Im}\, q(x,t)$

$\alpha = \sqrt{3/8}, \;\; \omega = -13/8$

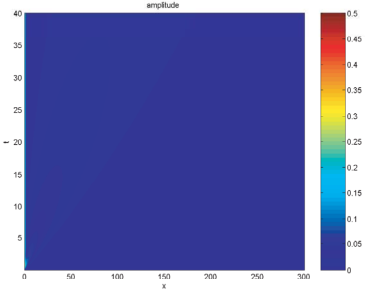
$q_0(x) \equiv 0, \; g_0(t) = \alpha \mathrm{e}^{2\mathrm{i}\omega t} + \mathrm{O}(\mathrm{e}^{-10t^2})$

Numerics for $\omega < -3\alpha^2$, II

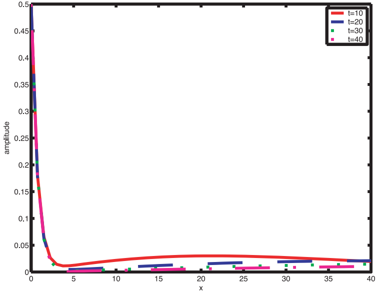
Numerical solution for $t = 20$, $0 < x < 100$.
Upper: real part $\mathrm{Re}\, q(x, 20)$. Lower: imaginary part $\mathrm{Im}\, q(x, 20)$.



Numerics for $\omega \geq \alpha^2/2$



Amplitude of $q(x,t)$

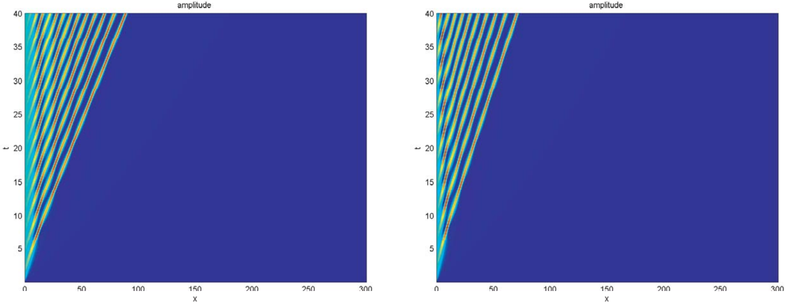


Amplitude for $t = 10, \ldots$

$\alpha = 0.5, \;\; \omega = 1, \;\; \omega \geq \alpha^2/2,$

$q_0(x) \equiv 0, \; g_0(t) = \alpha e^{2i\omega t} + O(e^{-10t^2})$

Amplitude of $q(x,t)$



$\alpha = 0.5$
$\omega = -2\alpha^2 = -0.5$

$\alpha = 0.5$
$\omega = -\alpha^2 = -0.25$

$$q_0(x) \equiv 0, \qquad g_0(t) = \alpha\, \mathrm{e}^{2i\omega t} + \mathrm{O}(\mathrm{e}^{-10t^2})$$

General scheme for boundary value problems via IST

Goal: adapt (generalize) the RHP approach to boundary-value (initial-boundary value) problems for integrable equations.

Data for an IBV problem (e.g, in domain $x > 0$, $t > 0$):

(i) Initial data: $q(x, 0) = q_0(x)$, $x > 0$

(ii) Boundary data: $q(0, t) = g_0(t)$, $q_x(0, t) = g_1(t), \ldots$.

Question: How many boundary values?

For a well-posed problem: roughly "half" the number of x-derivatives.

For NLS: one b.c. (e.g., $q(0, t) = g_0(t)$).

General idea for IBV: use both equations of the Lax pair as spectral problems.

Common difficulty: spectral analysis of the t-equation on the boundary ($x = 0$) involves more functions (boundary values $q(0, t), q_x(0, t), \ldots$) than possible data for a well-posed problem.

Half-line problem for NLS

For NLS: t-equation

$$\Phi_t = \left(-2\mathrm{i}k^2\sigma_3 + 2k\begin{pmatrix} 0 & q(x,t) \\ \lambda\bar{q}(x,t) & 0 \end{pmatrix} + \begin{pmatrix} -\mathrm{i}\lambda|q|^2 & \mathrm{i}q_x \\ -\mathrm{i}\lambda\bar{q}_x & \mathrm{i}\lambda|q|^2 \end{pmatrix}\right)\Phi$$

involves q and q_x; hence for the (direct) spectral analysis at $x=0$ one needs $q(0,t)$ and $q_x(0,t)$. Assume that we are given the both. Then one can define two sets of spectral functions coming from the spectral analysis of x-equation and t-equation.

1. $q(x,0) \mapsto \{a(k), b(k)\}$ (direct problem for x-equ); $\quad s \equiv \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$

   $\{q(0,t), q_x(0,t)\} \mapsto \{A(k), B(k)\}$ (direct problem for t-equ)
2. From the spectral functions $\{a(k), b(k), A(k), B(k)\}$, the jump matrix $J(x,t,k)$ for the Riemann-Hilbert problem is constructed:

   $$\{a(k), b(k), A(k), B(k)\} \mapsto J_0(k)$$
   $$J(x,t,k) = \mathrm{e}^{-\mathrm{i}(2k^2t+kx)\sigma_3} J_0(k) \mathrm{e}^{\mathrm{i}(2k^2t+kx)\sigma_3}$$

   (notice the same explicit dependence on (x,t)!) Jump conditions are across a contour Σ determined by the asymptotics of $g_0(t)$ and $g_1(t)$
3. Similarly to the Cauchy (whole-line) problem, the solution of the IBV (half-line) problem is given in terms of the solution of the RHP: $q(x,t) = 2\mathrm{i}\lim_{k\to\infty}(kM_{12}(x,t,k))$.

Given $q(x,t)$, how to construct $M(x,t,k)$?

Define $\Psi_j(x,t,k)$, $j = 1,2,3$ solutions (2×2) of the Lax pair equations specified at all "corners" of the (x,t)-domain where the IBV problem is formulated:

1. $\Psi_1(0,T,k) = \mathrm{e}^{-2\mathrm{i}k^2 T\sigma_3}$ $(\Psi_1(0,t,k) \simeq \mathrm{e}^{-2\mathrm{i}k^2 t\sigma_3}$ as $t\to\infty)$
2. $\Psi_2(0,0,k) = I$
3. $\Psi_3(x,0,k) \simeq \mathrm{e}^{-\mathrm{i}kx\sigma_3}$ as $x\to\infty$

Being simultaneous solutions of x-and t-equation, they are related by two scattering relations:

1. $\Psi_3(x,t,k) = \Psi_2(x,t,k)s(k)$ $\qquad s = \begin{pmatrix} \bar a & b \\ -\bar b & a \end{pmatrix}$
2. $\Psi_1(x,t,k) = \Psi_2(x,t,k)S(k;T)$ $\qquad S = \begin{pmatrix} \bar A & B \\ -\bar B & A \end{pmatrix}$

Then M is constructed from columns of Ψ_1, Ψ_2 and Ψ_3 following their analyticity and boundedness properties w.r.t k, and the jump relation for RHP is re-written scattering relations (i)+(ii) for Ψ_j.

For NLS in half-strip $(T<\infty)$ or in quarter plane $(T=\infty)$ with $g_j(t)\to 0$ as $t\to\infty$: first column of $\Psi_1(x,t,k)\mathrm{e}^{(-\mathrm{i}kx-2\mathrm{i}k^2t)\sigma_3}$ is bounded in $\{k : \mathrm{Im}\, k \geq 0, \mathrm{Im}\, k^2 \leq 0\}$, etc., which leads to $\Sigma = \mathbb{R}\cup \mathrm{i}\mathbb{R}$.

- Given $q_0(x)$, determine $a(k)$, $b(k)$: $\boxed{a(k) = \Phi_2(0,k), \quad b(k) = \Phi_1(0,k)}$, where vector $\Phi(x,k)$ is the solution of the x-equation evaluated at $t = 0$:

$$\Phi_x + \mathrm{i}k\sigma_3\Phi = Q(x, 0, k)\Phi, \quad 0 < x < \infty, \operatorname{Im} k \geq 0$$

$$\Phi(x,k) = e^{\mathrm{i}kx}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1)\right) \text{ as } x \to \infty,$$

$$Q(x,0,k) = \begin{pmatrix} 0 & q_0(x) \\ -\bar{q}_0(x) & 0 \end{pmatrix}$$

- Given $\{g_0(t), g_1(t)\}$, determine $A(k;T), B(k;T)$:

$$\boxed{A(k;T) = \mathrm{e}^{2\mathrm{i}k^2T}\overline{\tilde{\Phi}_1(T,\bar{k})}, \quad B(k;T) = -\mathrm{e}^{2\mathrm{i}k^2T}\tilde{\Phi}_2(T,k)},$$

where vector $\tilde{\Phi}(x,k)$ is the solution of the t-equation evaluated at $x = 0$:

$$\tilde{\Phi}_t + 2\mathrm{i}k^2\sigma_3\tilde{\Phi} = \tilde{Q}(0, t, k)\tilde{\Phi}, \quad 0 < t < T,$$

$$\tilde{\Phi}(0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\tilde{Q}(0,t,k) = \begin{pmatrix} -|g_0(t)|^2 & 2kg_0(t) - \mathrm{i}g_1(t) \\ 2k\bar{g}_0(t) + \mathrm{i}\bar{g}_1(t) & |g_0(t)|^2 \end{pmatrix}$$

- Contour: $\Sigma = \mathbb{R} \cup \mathrm{i}\mathbb{R}$
- Jump matrix: $J(x,t,k) = \mathrm{e}^{-(\mathrm{i}kx + 2\mathrm{i}k^2 t)\sigma_3} J_0(k) \mathrm{e}^{(\mathrm{i}kx + 2\mathrm{i}k^2 t)\sigma_3}$ with

$$J_0(k) = \begin{cases} \begin{pmatrix} 1 + |r(k)|^2 & \bar{r}(k) \\ r(k) & 1 \end{pmatrix}, & k > 0, \\ \begin{pmatrix} 1 & 0 \\ C(k;T) & 1 \end{pmatrix}, & k \in \mathrm{i}\mathbb{R}_+, \\ \begin{pmatrix} 1 & \bar{C}(\bar{k};T) \\ 0 & 1 \end{pmatrix}, & k \in \mathrm{i}\mathbb{R}_-, \\ \begin{pmatrix} 1 + |r(k) + C(k;T)|^2 & \bar{r}(k) + \bar{C}(k;T) \\ r(k) + C(k;T) & 1 \end{pmatrix}, & k < 0, \end{cases}$$

where $r(k) = \dfrac{\bar{b}(k)}{a(k)}$, $C(k;T) = -\dfrac{\overline{B(\bar{k};T)}}{a(k)d(k;T)}$ with $d = a\bar{A} + b\bar{B}$

(also works for $T = +\infty$ if $g_0(t), g_1(t) \to 0$, $t \to \infty$)

Compatibility of boundary values: Global Relation

- The fact that the set of initial and boundary values $\{q_0(x), g_0(t), g_1(t)\}$ cannot be prescribed arbitrarily (as data for IBVP) must be reflected in spectral terms.

  Indeed, from scattering relations (i)+(ii):
  $S^{-1}(k;T)s(k) = \Psi^{-1}(x,t,k)\Psi_3(x,t,k)$. Evaluating this at $x=0$, $t=T$ and using analyticity and boundedness properties of Ψ_j, one deduces for the (12) entry of $S^{-1}s$:

$$\boxed{A(k;T)b(k) - a(k)B(k;T) = O\left(\frac{\mathrm{e}^{4\mathrm{i}k^2T}}{k}\right), \ k \to \infty}$$

$$k \in D = \{\operatorname{Im} k \geq 0, \operatorname{Re} k \geq 0\}$$

- This relation is called Global Relation (GR): it characterizes the compatibility of $\{q_0(x), g_0(t), g_1(t)\}$ in spectral terms.

Typical theorem: Consider the IBVP with given $q_0(x)$ and $g_0(t)$. Assume $g_1(t)$ is such that the spectral functions $\{a(k), b(k), A(k), B(k)\}$ calculated from $\{q_0(x), g_0(t), g_1(t)\}$ satisfy Global Relation. Then the solution of the IBVP is given in terms of the solution of the RHP above. Moreover, it satisfies also the b.c. $q_x(0,t) = g_1(t)$.

Resolving Global Relation (GR) in linear case $iq_t + q_{xx} = 0$

1. construct the Dirichlet-to-Neumann map $\{q_0(x), g_0(t)\} \mapsto g_1(t)$:

$$
\begin{aligned}
g_1(t) = &-\frac{\mathrm{i}}{\pi}\int_{\partial D} \mathrm{d}k \mathrm{e}^{-\mathrm{i}k^2 t} k \left(\int_0^\infty q_0(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x\right) \\
&+ \frac{1}{\pi}\int_{\partial D} \mathrm{d}k \left\{\mathrm{i}k^2 \int_0^t \mathrm{e}^{\mathrm{i}k^2(\tau - t)} g_0(\tau)\,\mathrm{d}\tau - g_0(t)\right\}
\end{aligned}
$$

2. solve the IBVP:

$$
\begin{aligned}
q(x,t) = &\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx - \mathrm{i}k^2 t}(\hat{q}_0(k) - \hat{q}_0(-k))\,\mathrm{d}k \\
&- \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}k \mathrm{e}^{-\mathrm{i}kx - \mathrm{i}k^2 t} k \left(\int_0^t \mathrm{e}^{\mathrm{i}k^2\tau} g_0(\tau)\right)
\end{aligned}
$$

where $\hat{q}_0(k) = \int_0^\infty \mathrm{e}^{\mathrm{i}kx} q_0(x)\,\mathrm{d}x$.

Using Global Relation for NLS

GR can also be used to describe the Dirichlet-to-Neumann map:

$$g_1(t) = \frac{g_0(t)}{\pi}\int \mathrm{e}^{-2\mathrm{i}k^2t}\left(\tilde{\Phi}_2(t,k) - \tilde{\Phi}_2(t,-k)\right)dk + \frac{4\mathrm{i}}{\pi}\int \mathrm{e}^{-2\mathrm{i}k^2t} kr(k)\overline{\tilde{\Phi}_2(t,\bar{k})}dk$$
$$+ \frac{2\mathrm{i}}{\pi}\int \mathrm{e}^{-2\mathrm{i}k^2t}(k[\tilde{\Phi}_1(t,k) - \tilde{\Phi}_1(t,-k)] + \mathrm{i}g_0(t))dk \quad \left(\int = \int_{\partial D}\right)$$

But: nonlinear! (g_1 is involved in the construction of $\tilde{\Phi}_j$)

- In the small-amplitude limit, this reduces to a formula giving $g_1(t)$ in terms of $g_0(t)$ and $q_0(x)$ (via $r(k)$); here NLS reduces to a linear equation $\mathrm{i}q_t + q_{xx} = 0$.
- This suggests perturbative approach: given $g_0(t)$ say periodic with small amplitude, derive a perturbation series for $g_1(t)$, with periodic terms.

For $T = \infty$: the approach can be implemented for boundary values non-decaying as $t \to \infty$. But for this: one needs correct large-time behavior of $g_1(t)$ complying with that of the given $g_0(t)$; this is because both $g_0(t)$ and $g_1(t)$ determine the spectral problem for t-equation and thus the structure of associated spectral functions $A(k)$, $B(k)$.

Dirichlet-to-Neumann map

Let $q(0,t)=\alpha\,\mathrm{e}^{2\mathrm{i}\omega t}$ $(q(0,t)-\alpha\,\mathrm{e}^{2\mathrm{i}\omega t}\to 0,\ t\to\infty)$
Neumann values $(q_x(0,t))$:

- from numerics:

$$q_x(0,t)\simeq c\,\mathrm{e}^{2\mathrm{i}\omega t} \qquad c=\begin{cases}2\mathrm{i}\alpha\sqrt{\frac{\alpha^2-\omega}{2}}, & \omega\le -3\alpha^2\\ \alpha\sqrt{2\omega-\alpha^2}, & \omega\ge\frac{\alpha^2}{2}\end{cases}$$

- theoretical (asymptotic) results: agree with numerics (for all $x>0,\ t>0$) provided c as above.

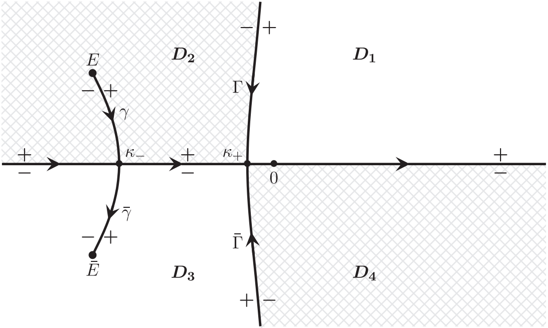
Question: Why these particular values of c?

(the spectral mapping $\{g_0,g_1\}\mapsto\{A(k),B(k)\}$ is well-defined for any $c\in\mathbb{C}$!)

Idea: Use the global relation (its impact on analytic properties of $A(k)$, $B(k)$) to specify admissible values of parameters α,ω,c.

The RHP for NLS: the contour

for $\omega < -3\alpha^2$, assuming $q_x(0,t) \sim 2\mathrm{i}\alpha\beta\,\mathrm{e}^{2\mathrm{i}\omega t}$



$\Sigma = \mathbb{R} \cup \gamma \cup \bar{\gamma} \cup \Gamma \cup \bar{\Gamma}$ with $E = -\beta + \mathrm{i}\alpha$.

The RHP for NLS: the jump matrix

$$J(x,t;k) = \begin{cases} \begin{pmatrix} 1 & -\bar{\rho}(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ -\rho(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1+|\rho(k)|^2 \end{pmatrix} & k \in (-\infty, \kappa_+), \\ \begin{pmatrix} 1 & -\bar{r}(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ -r(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1+|r(k)|^2 \end{pmatrix} & k \in (\kappa_+, \infty), \\ \begin{pmatrix} 1 & 0 \\ c(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix} & k \in \Gamma, \\ \begin{pmatrix} 1 & \bar{c}(\bar{k})\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix} & k \in \bar{\Gamma}, \\ \begin{pmatrix} 1 & 0 \\ f(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix} & k \in \gamma, \\ \begin{pmatrix} 1 & -\bar{f}(\bar{k})\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix} & k \in \bar{\gamma}. \end{cases}$$

where $\boxed{\theta(k) = \theta(k,\xi) = 2k^2 + 4\xi k}$ with $\boxed{\xi = \frac{x}{4t}}$

Literature

I. Integrable nonlinear PDE; Riemann-Hilbert problem as a tool

- T.Trogdon and S.Olver, Riemann-Hilbert problems, their numerical solutions, and the computation of nonlinear special functions, SIAM, Philadelphia, 2012.
- P.Deift, Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach. Courant Lecture Notes in Mathematics, New York, AMS, 2000.
- L.D.Faddeev and L.Takhtajan, Hamiltonian Methods in the Theory of Solitons, Springer, 1987.
- M.J.Ablowitz and H.Segur, Solitons and the Inverse Scattering Transform, Philadelphia, PA, SIAM, 1981.

II. Problems with step-like initial data

- A.Boutet de Monvel, V.Kotlyarov and D.Shepelsky, Focusing NLS equation: long-time dynamics of step-like initial data, International Mathematics Research Notices (2011), no. 7, 1613-1653.
- G. Biondini, S. Li and D. Mantzavinos, Soliton trapping, transmission and wake in modulationally unstable media, Phys. Rev. E (2018), arXiv:1810.00388.
- G. Biondini and D. Mantzavinos, Long-time asymptotics for the focusing nonlinear Schrödinger equation with nonzero boundary conditions at infinity and asymptotic stage of modulational instability, arXiv:1512.06095
- I. Egorova, Z. Gladka, V. Kotlyarov, and G. Teschl, Long-Time Asymptotics for the Korteweg-de Vries Equation with Steplike Initial Data, Nonlinearity 26, 1839-1864 (2013).
- K. Andreiev, I. Egorova, T. L. Lange, and G. Teschl, Rarefaction Waves of the Korteweg-de Vries Equation via Nonlinear Steepest Descent, J. Differential Equations 261, 5371-5410 (2016).