

Value Statistics of Chaotic Wigner Function

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We study Wigner function value statistics of classically chaotic quantum maps on compact 2D phase space. We show that the Wigner function statistics of a random state is a Gaussian, with the mean value becoming negligible compared to the width in the semi-classical limit. Using numerical example of quantized sawtooth map we demonstrate that the relaxation of time-dependent Wigner function statistics, starting from a coherent initial state, takes place on a logarithmically short ($\propto \log \hbar$) time scale.

The Wigner function¹⁾ (WF) is an essential concept of phase space representation of quantum mechanics, namely it is a useful and faithful representation of a pure or mixed quantum state in terms of functions of canonical classical phase space variables. It has many applications in various branches of physics, in particular in quantum optics. However, WF cannot be interpreted as quantum phase space distribution as it can develop negative values, in particular due to well known oscillatory interference fringes following, e.g., coherent wave-packet superpositions.²⁾ WF played an important role in the realm of quantum chaology.^{3),4)} In particular, it has been conjectured that WF of stationary eigenstates of bounded dynamical systems in quasi-classical regimes condense, within classical scales, onto classically invariant components of phase space. On the other hand, within a smaller-quantum scale, e.g. of Planck cell size, the phase space structure of WF of typical, say random or ergodic states, is very much unknown. This question is important for the understanding of decoherence and quantum stability with respect to system's perturbations as discussed recently.^{2),5)-7)}

In this paper we address the question of the structure of typical WF from a statistical point of view. We define and analyze the statistical distribution of values of WF of a given quantum state. In particular, we are interested in the relaxation of this distribution with time, when we start from initial coherent state, and in the corresponding time scale. Here we limit ourselves on classically chaotic discrete time systems, namely chaotic quantum maps, where the full phase space is the only topologically transitive classical ergodic component.

We show that for fully chaotic quantum maps the limiting WF value distribution is a Gaussian. The average value is fixed by normalization, while the second moment (or the variance) diverges in the quasi-classical limit ($\hbar \rightarrow 0$), so we have roughly a symmetric distribution of positive and negative values of WF for random states. In addition, we show that for chaotic systems, the relaxation to equilibrium in a statistics of WF happens on a short $\propto \log \hbar$ (Ehrenfest) time scale.

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We consider Wigner functions^{1),3)} of a pure state $|\psi\rangle$ from the Hilbert space corresponding to a classical system with a compact phase space χ . For explicit calculation we choose 2D toroidal phase space $\chi = \mathbb{T}_2$ that we quantize following Ref. 8). Let us write canonical Hilbert space basis of position states as $|n\rangle$, $n = 0, \dots, N-1$. Due to technical reasons N is *odd*. WF is defined on a regular grid of $N \times N$ phase-space points

$$W_\psi(n, m) = \frac{N}{\sqrt{N-1}} \sum_{n', l} e^{-2i\pi n' m/N} \tilde{\delta}_{2l-2n+n'} \langle l+n' | \psi \rangle \langle \psi | l \rangle, \quad \tilde{\delta}_k = \frac{1}{N} \sum_{m'} e^{i\pi m' k/N},$$

where all indices run from $-(N-1)/2$ to $(N-1)/2$, as will be assumed whenever we refer to the quantized torus. We study WF value distribution $P(w)$ defined as

$$P(w) = \frac{1}{V} \int_\chi \delta(w - W(x)) dx, \quad (1)$$

where V is the phase space volume (area). In our case: $(1/V) \int_\chi = (1/N^2) \sum_{m,n}$. Let $\overline{(\dots)}$ denote the average with respect to $P(w)$. The first two moments of $P(w)$, the mean value \overline{W} and the standard deviation σ , are determined by the normalization of probability and purity $\rho^2 = \rho$ of the state $\rho = |\psi\rangle\langle\psi|$. We shall ‘normalize’ WF by unitarizing the standard deviation

$$\sigma^2 = 1 \quad \text{and} \quad \overline{W} = (N-1)^{-1/2}. \quad (2)$$

Particularly interesting is the *relative standard deviation* defined as $\kappa = \sigma/\overline{W}$ that scales with dimension N as $\kappa \sim \sqrt{N}$, i.e. it *diverges* in the semi-classical limit.

The states of developed quantum chaos are associated with ergodic or random wave functions⁹⁾ that can be defined in Hilbert space basis $|l\rangle$, $l = 0, \dots, N-1$ as

$$|\psi\rangle = \sum_{l=1}^N c_l |l\rangle, \quad \sum_{l=1}^N |c_l|^2 = 1, \quad (3)$$

where $c_l \in \mathbb{C}$ are random coefficients, that may be considered for large N as independent Gaussian variables¹⁰⁾ of variance $\langle |c_l|^2 \rangle = 1/N$. $\langle \dots \rangle$ denotes ensemble average over random states (3). WF value statistics of a random state at fixed point in phase space should be transitionally invariant and thus equal to the expected statistics of a ‘chaotic Wigner function’. Therefore we can compute $P(w)$ by analyzing the distribution of WF values at one phase space point, e.g., $W(0, 0)$,

$$W(0, 0) = \frac{N}{\sqrt{N-1}} \left(|c_0|^2 + \sum_{l \neq l'} \tilde{\delta}(l+l') c_l^* c_{l'} \right). \quad (4)$$

The expression (4) is a sum of N statistically independent terms, where the first $|c_0|^2$ is strictly non-negative and others have vanishing mean and finite fluctuation. The central limit theorem, for large $N \gg 1$, implies that the distribution of $W(0, 0)$ is a Gaussian

$$P_{\text{random}}(w) = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} (w - \overline{W})^2 \right), \quad (5)$$

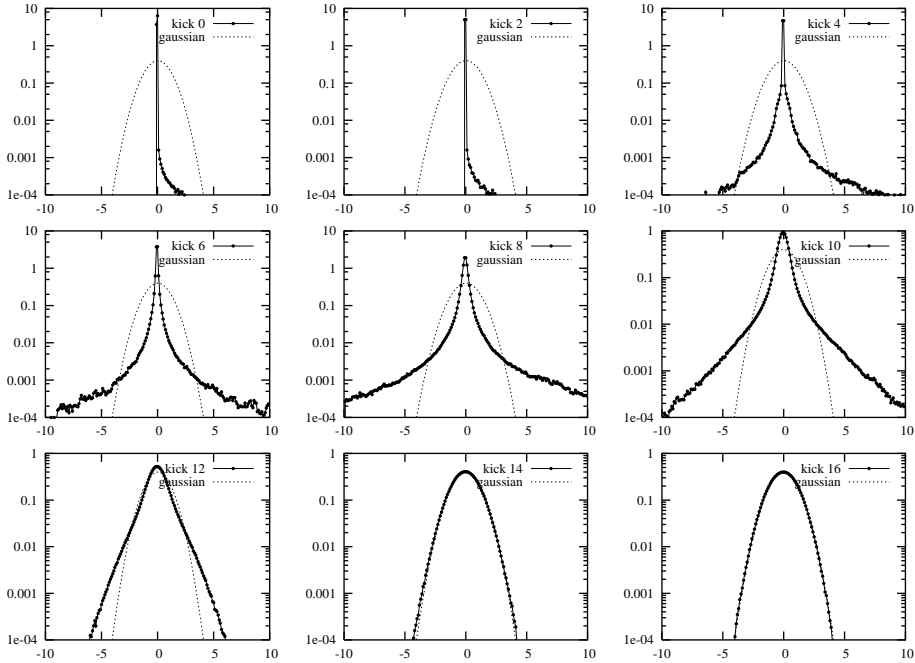


Fig. 1. Time evolution of the WF value distribution (1) of the quantized sawtooth map for $K = 0.5$ and $N = 2187$. Initial state is a coherent wave packet. Dashed curves give a theoretical Gaussian (5).

with the first and the second moment equal to those of WF value distribution over the whole phase space (2).

The dynamics is studied on the generic example of the quantized saw-tooth map on the toroidal phase space. Quantized sawtooth map with the kicking strength K is defined by the following evolution operator¹¹⁾

$$U = \exp\left(-i\frac{T}{2}\hat{p}^2\right) \exp\left(i\frac{KT}{2}\hat{n}^2\right),$$

where $\hat{n}|n\rangle = n|n\rangle$, $\exp(2\pi i\hat{p}/N)|n\rangle = |n+1\rangle$, and $T = 2\pi/N$. Here we consider the regime $K > 0$ for which the classical sawtooth map is ergodic and uniformly hyperbolic. In Fig. 1 we show distribution $P(w)$ of time-evolving state in the sawtooth map for $K = 0.5$ and $N = 2187$, starting from a coherent state. We see that $P(w)$ quickly relaxes from a particular initial form to the Gaussian distribution (5) predicted for random waves. The relaxation is the result of the wave self-interference caused by the corresponding classical stretching of the wave packet over the phase space.

To measure the deviation of distribution $P(w)$ from the Gaussian statistics we define the excess ϵ of the distribution by formula

$$\epsilon = \overline{(w - \overline{W})^4} / \sigma^4 - 3. \quad (6)$$

Note, that $\epsilon = 0$ for a Gaussian (5) and the size of ϵ measures a deviation from Gaussian. Using the excess ϵ we can quantitatively measure the transition of statistics to

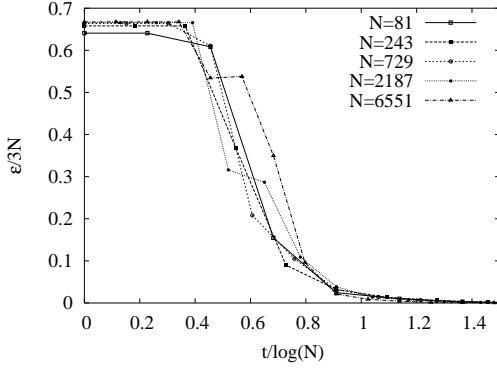


Fig. 2. The excess ϵ of the WF value distribution as a function of scaled time $t/\log N$ for several different values of N . The parameter of the quantized sawtooth map is $K = 0.5$, and the initial state is a coherent wave packet.

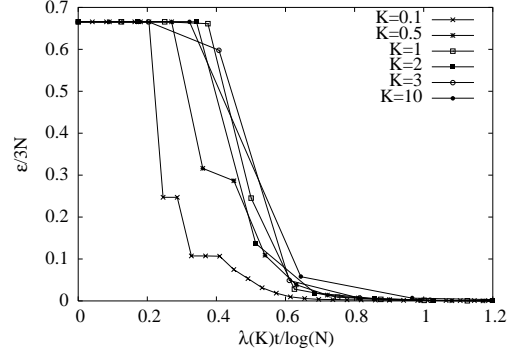


Fig. 3. Scaled time evolution of the excess ϵ for the quantized saw-tooth map at different kicking strengths K and fixed dimension $N = 2187$. The Lyapunov exponent is given by $\lambda(K) = \log((2 + K + \sqrt{(2 + K)^2 - 4})/2)$.¹¹⁾

the asymptotic Gaussian form, that happens on the Ehrenfest time scale

$$t_r \sim \log(N)/\lambda, \quad (7)$$

where λ is an effective Lyapunov exponent and N is the dimension of Hilbert space. Numerical experiments for the family of sawtooth maps show that t_r (7) has the right scaling, both with N (see Fig. 2) and with λ (see Fig. 3). To check the rate of breakdown of direct classical-quantum correspondence we propose to study the percentage of phase space with negative valued WF

$$P_- = \int_{-\infty}^0 P(w)dw \quad (8)$$

as a function of time. For random states P_- goes to $1/2$ in the limit $N \rightarrow \infty$. In Fig. 4 we show the evolution of P_- in the quantized saw-tooth map, initially in a coherent state, for different dimensions N . We see that P_- converges very fast to $1/2$ on a time scale t_c proportional to $\log(N)$ but significantly smaller than relaxation time t_r . However, both time scales have the same scaling with N and λ .

We have studied WF statistics of random states, and of time-dependent states of time-periodic, e.g. kicked quantum systems. The statistical approach to value statistics is meaningful since two-point phase space correlations of WF vanish (for distinct points) in the semi-classical limit as shown in a longer paper.¹³⁾ We have proven that random states corresponding to the regime of quantum chaos have Gaussian WF value statistics, with the relative standard deviation divided by the mean value diverging as $\sigma/\overline{W} \sim \sqrt{N}$ in the semi-classical limit $N \rightarrow \infty$. In addition we have shown that WF statistics in a classically fully chaotic system, initially in coherent state, tends to a Gaussian statistics within the Ehrenfest time scale t_r . We believe that our results may shed some light onto the related studies of decoherence^{2),12)} and the fidelity of two initially equivalent states which undergo two slightly different

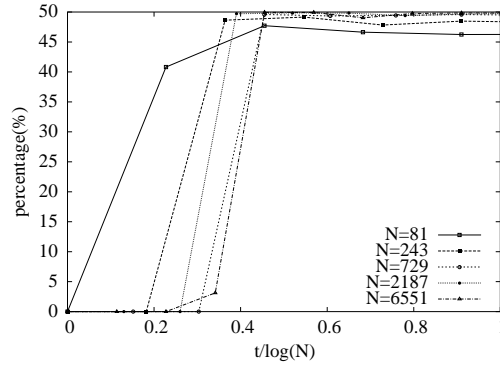


Fig. 4. Percentage of phase space with negative WF as a function of scaled time for the quantized saw-tooth map, $K = 0.5$, at several different dimensions N .

time-evolutions.⁵⁾ The randomness of WFs in classically chaotic systems suggests a description by means of random functions. The latter is the subject of the more detailed publication 13).

Acknowledgements

Useful discussions with A. Bäcker, G. Veble and M. Žnidarič, as well as the financial support by the Ministry of Education, Science and Sport of Slovenia are gratefully acknowledged.

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