

The two-body random spin ensemble and a new type of quantum phase transition

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New Journal of Physics **10** (2008) 023020 (13pp)

Received 7 November 2007

Published 15 February 2008

Online at <http://www.njp.org/>

doi:10.1088/1367-2630/10/2/023020

Abstract. We study in this paper the properties of a two-body random matrix ensemble for distinguishable spins. We require the ensemble to be invariant under the group of local transformations and analyze a parametrization in terms of the group parameters and the remaining parameters associated with the ‘entangling’ part of the interaction. We then specialize to a spin chain with nearest-neighbour interactions and numerically find a new type of quantum-phase transition related to the strength of a random external field, i.e. the time-reversal-breaking one-body interaction term.

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1. Introduction

Eugene Wigner introduced random matrix models to nuclear physics about 50 years ago [1]. The scope of their applications has increased over the years [2, 3], including fields such as molecular and atomic physics, mesoscopics and field theory. More recently, random matrix theory has started to be used in quantum information theory [4]–[8]. For an introduction to such applications, see [9]. There, the concept of individual qubits and their interactions becomes important. This implies that we enter the field of two-body random ensembles (TBRE) [10, 11], i.e. ensembles of Hamiltonians of n -body systems interacting by two-body forces. While such ensembles have received considerable attention, the focus was first on fermions and later also included bosons. Yet in quantum information theory, the qubits are taken to be distinguishable, and indeed the same holds for spintronics. Recently, interest in both fields has increased sharply [12, 13].

It is thus very pertinent to formulate and investigate TBREs for distinguishable qubits. As random matrix ensembles are mainly determined by their symmetry properties, this ensemble will be very different from other TBREs. In particular, as the particles are distinguishable, their interaction can vary from one particle pair to another and can indeed be randomly distributed, thus introducing an entirely new aspect. This has the consequence that the topology according to which spins or qubits are distributed or interact would be important. Thus chains, trees and crystals of particles with nearest, second nearest and up to k th order interactions can be represented.

As mentioned above, random matrix ensembles are usually basically defined by the invariance group of their measure and, if that is not enough, some minimal information conditions [9, 14] or independence condition [15]. Note that we deal with the symmetry of the ensemble, rather than with the symmetry of individual systems. The two concepts are, to some degree, complementary, and the former has also been called *structural invariance*. We propose an adequate definition for such ensembles in a very general framework in terms of independent Gaussian distributed variables. We then give an alternative representation in terms of the invariance group and variables that determine the orbits of the Hamiltonian on the ensemble under the action of the group.

In order to show the relevance of the new ensemble, we address the simplest possible topology, namely a chain with nearest-neighbour interactions. For this system, we focus on the ensemble-averaged structure of the ground state and demonstrate the existence of an unusual quantum-phase transition [16], which is triggered by breaking of *time-reversal invariance* (TRI).

Entanglement, a key resource of quantum many-body systems in terms of quantum information, is to a large extent related to quantum correlations, localization properties and quantum chaos. Entanglement has also been used as a property, an alternative to long-range order in spatial correlation functions, to describe systems undergoing a quantum phase transition [17]. In one-dimensional (1D) systems, such as quantum spin chains, it has been shown [18] that the entanglement entropy of the ground state typically saturates or diverges logarithmically with size when approaching the thermodynamic limit. Furthermore, it has been shown that logarithmic divergence implies quantum criticality.

Interesting results emerge when a spatially homogeneous spin model is replaced by its disordered counterpart, where the spin interactions are taken at random. In this case, there is often no physical justification why random interactions should still obey specific restricted forms such as Ising or Heisenberg interactions. In this context, we argue that it is more natural to use two-spin random ensembles (TSRE) for distinguishable particles, specifically choosing quantum spins $1/2$, although these ensembles can readily be generalized to arbitrary spin. By construction, these ensembles, as given in section 2, are invariant with respect to arbitrary local rotations, which we may view as gauge transformations. Another physical motivation for the definition of such ensembles is the coupling among arbitrary and perhaps mutually independent two-level quantum systems, which may come from completely different physical contexts, such as two-level atoms, Josephson junctions and photons.

In section 3, we concentrate on 1D systems or spin chains and present the results of numerical calculations, mainly based on density matrix renormalization group (DMRG) [19], in which we investigate entanglement and correlation properties of the ground state, averaged over an ensemble, and the average spectral gap to the first excited state as well as its fluctuations. If we include the interaction with an external random magnetic field and, hence, TRI is broken, we find fast decay of correlations, saturation of entanglement entropy, and power law decay $g \approx N^{-0.4}$ of the spectral gap g with the system size N , while its distribution displays Wigner-type level repulsion. When the strength of an external field tends to zero and TRI is restored, we find long-range order, logarithmically divergent entanglement entropy and *exponential* decay of the spectral gap, while the level repulsion disappears.

We argue that this quantum phase transition is non-conventional from the point of view of established models, since in what we shall call the non-critical case we still find slow power law closing of the spectral gap.

2. The embedded ensemble of spin Hamiltonians with random two-body interactions

In this section, we define the TSRE of Hamiltonians for systems with N distinguishable spins or qubits with at the most two-body interactions and describe its basic (invariance) properties. If we do not allow all spins to interact, we have to define which ones do. The simplest case will be a chain with nearest-neighbour interactions, but in general we need a graph whose vertices correspond to spins and whose edges correspond to two-body interactions. We proceed to formalize this.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an *undirected graph* with a finite set of N vertices \mathcal{V} and a set of M edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. In addition, let $\lambda : \mathcal{V} \rightarrow \mathbb{R}^+$ and $\mu : \mathcal{E} \rightarrow \mathbb{R}^+$ be two non-negative functions defined on the sets of vertices and edges, respectively. To such a graph we assign a 2^N -dimensional Hilbert space of N spins or qubits, placed at its vertices $\mathcal{H}_{\mathcal{G}} = \otimes_{j \in \mathcal{V}} \mathbb{C}^2 \equiv \mathbb{C}^{2^N}$, and a set of N Pauli operators $\sigma_j^\alpha : \mathcal{H}_{\mathcal{G}} \rightarrow \mathcal{H}_{\mathcal{G}}$, $\alpha \in \{1, 2, 3\}$, $j \in \mathcal{V}$ satisfying $\mathcal{SO}(3)$ commutation relations $[\sigma_j^\alpha, \sigma_k^\beta] = i\varepsilon_{\alpha\beta\gamma} \sigma_j^\gamma \delta_{j,k}$. We also use the notation $\vec{\sigma}_j = (\sigma_j^1, \sigma_j^2, \sigma_j^3)$.

Let $A^{(j,k)} \in \mathbb{R}^{3 \times 3}$, $(j, k) \in \mathcal{E}$, be a set of M *random* real 3×3 matrices, and $\vec{b}^{(j)} \in \mathbb{R}^3$, $j \in \mathcal{V}$, be a set of N *random* 3D real vectors. The TSRE then consists of the *random* Hamiltonians

$$H = \sum_{(j,k) \in \mathcal{E}} \mu(j, k) \vec{\sigma}_j \cdot A^{(j,k)} \vec{\sigma}_k + \sum_{j \in \mathcal{V}} \lambda(j) \vec{b}^{(j)} \cdot \vec{\sigma}_j. \quad (1)$$

The above-defined functions on the edges and vertices of the graph are used to determine the average strength of the corresponding terms in the Hamiltonian. The distribution of random two-body interaction matrices (for brevity, also called *bond matrices*) $A^{(j,k)}$ and the random external field vectors $\vec{b}^{(j)}$ shall be uniquely determined by requiring the following two conditions: *maximum local invariance* and *maximum independence* expressed formally as:

- (i) An ensemble of Hamiltonians (1) should be invariant with respect to an arbitrary *local* $\mathcal{SO}(3)$ transformation, namely

$$\vec{\sigma}'_j = O_j \vec{\sigma}_j, \quad (2)$$

where $O_j \in \mathcal{SO}(3)_j$, $j \in \mathcal{V}$, meaning that the choice of a local coordinate system is arbitrary for each spin/qubit. Obviously, (2) preserves the canonical commutation relations for the Pauli operators. Then, it follows immediately that the *joint probability distributions* of $\{A^{(j,k)}, \vec{b}^{(j)}\}$ should be invariant with respect to transformations

$$A^{(j,k)'} = O A^{(j,k)} O', \quad \vec{b}^{(j)'} = O \vec{b}^{(j)}, \quad (3)$$

where O, O' are arbitrary independent $\mathcal{SO}(3)$ rotations for each j, k .

- (ii) The matrix elements of the tensors $A_{\alpha,\beta}^{(j,k)}$ and of the vectors $\vec{b}_\alpha^{(j)}$ should be *independent* random variables.

Following arguments similar to those presented in [15], it is straightforward to show that, in order to satisfy conditions (i) and (ii) above for pre-determined but general strengths of bonds $\mu(j, k)$ and external fields $\lambda(j)$, $A_{\alpha,\beta}^{(j,k)}$ and $\vec{b}_\alpha^{(j)}$ should be *Gaussian independent random variables* of zero mean and equal variance, which are uniquely specified in terms of the correlators

$$\begin{aligned} \langle A_{\alpha,\beta}^{(j,k)} A_{\alpha',\beta'}^{(j',k')} \rangle &= \delta_{jj'} \delta_{kk'} \delta_{\alpha\alpha'} \delta_{\beta\beta'}, \\ \langle b_\alpha^{(j)} b_{\alpha'}^{(j')} \rangle &= \delta_{jj'} \delta_{\alpha\alpha'}, \\ \langle A_{\alpha,\beta}^{(j,k)} b_{\alpha'}^{(j')} \rangle &= 0, \end{aligned}$$

where $\langle \bullet \rangle$ denotes an ensemble average. We abbreviate the ensemble defined in this way by $\text{TSRE}(\mathcal{G}, \mu, \lambda)$ noting that it depends on the graph \mathcal{G} and the strength functions μ and λ .

Let us now describe some other elementary properties of TSRE. We have seen that each member H of $\text{TSRE}(\mathcal{G}, \mu, \lambda)$ can be parametrized (1) by $9M + 3N$ independent random parameters. However, we know, for the classical ensembles, that a parametrization in terms of

the structural invariance group, i.e. the invariance group of the ensemble and the remaining parameters, is very useful. Similarly, in the present case, for an arbitrary set of local $\mathcal{SO}(3)$ rotations O_j , the transformation of the parameters

$$A^{(j,k)'} = O_j^T A^{(j,k)} O_k, \quad \vec{b}^{(j)'} = O_j^T \vec{b}^{(j)}, \quad (4)$$

preserves the spectrum and all entanglement properties of H [20]. In fact, the transformation (4) can be considered as a *gauge transformation* since, composed with the local canonical transformation (2), it preserves the Hamiltonian H (1) exactly. Two Hamiltonians, specified by $\{A^{(j,k)}, b^{(j)}\}$ and $\{A^{(j,k)'}, b^{(j)'}\}$, can thus be considered equivalent, and the gauge transformation (4) defines a natural equivalence relation in $\text{TSRE}(\mathcal{G}, \mu, \lambda)$. Therefore, it may be of interest to consider the simplest parametrization of the set of equivalence classes, or in other words the orbits of a Hamiltonian on the ensemble under the structural invariance transformations. We thus ask: what is the general *canonical form* to which each element H can be brought by gauge transformations (4) and how can it be parametrized? Let the integer K denote the number of such parameters. The equivalent question for the classical ensembles leads to the eigenvalues as canonical parameters, since there the structural invariance group is much bigger.

Let us first consider the simplest connected graph, namely an open 1D chain of N vertices $\{1, \dots, N\}$ and $M = N - 1$ bonds. There, it turns out that the matrices $A^{(j,j+1)}$ can be simultaneously *symmetrized*, namely all $A^{(j,j+1)'} = [A^{(j,j+1)}]^T$, by choosing the following gauge transformation:

$$O_{j+1} = R_j O_j, \quad \text{where } R_j := V^{(j,j+1)} [U^{(j,j+1)}]^T \quad (5)$$

and where

$$A^{(j,j+1)'} =: U^{(j,j+1)} D^{(j,j+1)} [V^{(j,j+1)}]^T \quad (6)$$

is a standard canonical singular value decomposition (SVD) of the original bond matrix $A^{(j,j+1)}$, with $U^{(j,j+1)}, V^{(j,j+1)} \in \mathcal{SO}(3)$ and $D^{(j,j+1)}$ diagonal matrices of singular values. Since the initial transformation O_1 is still free, we can choose it as $O_1 := U^{(1,2)}$ such that the first bond matrix is even *diagonalized*, $A^{(1,2)'} = D^{(1,2)}$. This symmetrization is unique provided that singular values of all SVD's (6) are non-degenerate, which is the case for a generic member H . Thus the number of parameters specifying the bond matrices is $3 + 6(M - 1) = 6M - 3$ and, in addition to $3N$ external field parameters, this gives $K = 6M - 3 + 3N$ independent parameters. We recover the original set of parameters if we add the $3N$ parameters of the group of local rotations.

Secondly, we consider the case of a ring graph with N vertices and $M = N$ bonds, which is obtained from the previous case by specifying the periodic boundary condition $N + 1 \equiv N$. We see that a general H as given in (1) can now be symmetrized only if the additional topological condition $R := R_N \cdots R_2 R_1 = 1$ is satisfied. Now, all but one bond matrix can be symmetrized, for example, the last one may in addition be multiplied by a topological rotation $A^{(N,1)'} = A_{\text{symmetric}}^{(N,1)} R$ leading to three additional parameters. Therefore, in the case of a ring graph, we have $K = 6M + 3N$ independent parameters; again adding the $3N = 3M$ parameters of the local rotations, we obtain the full set of parameters.

We can now consider the case of a general (connected) graph. From the previous examples, it is evident that the only crucial additional parameter is the number L of *primitive* cycles, i.e. those cycles that cannot be decomposed into other primitive cycles. It is clear that each primitive cycle adds three additional topological parameters (or one $\mathcal{SO}(3)$ topological R matrix) to the $6M - 3 + 3N$ parameters which we would have for the case of a *tree* graph. Therefore we have

$$K = 6M + 3N + 3L - 3. \quad (7)$$

Counting the cycle-contributions and taking the primitivity criterion into account again, we obtain the the total number of parameters by adding those of the local rotations.

The above considerations hold for the case of general bond and vertex strength functions μ and λ . If, however, these functions are degenerate, or even constant, i.e. the average interaction strength and field strength do not depend on edges/vertices of the graph, then the structural invariance group of the TSRE may be even larger. In the latter case, this group is obtained as a semi-direct product of a *discrete symmetry group* of the graph \mathcal{G} and the gauge group of local rotations, the latter being the normal subgroup. We hope that the considerable invariance properties of the TSRE will prove useful in a future analytical treatment of its properties.

3. The properties of ground states of the TSRE on a 1D chain

In this section, we shall only consider the simplest case of a TSRE on a 1D chain of N vertices. In addition, we consider the most symmetric case of constant strength functions, say $\mu(j, j+1) \equiv 1$ and $\lambda(j) \equiv \lambda = \text{const}$. Such an ensemble of random spin chain Hamiltonians

$$H = \sum_{j=1}^{N-1} \vec{\sigma}_j \cdot A^{(j,j+1)} \vec{\sigma}_{j+1} + \lambda \sum_{j=1}^N \vec{b}^{(j)} \cdot \vec{\sigma}_j \quad (8)$$

shall be designated as $\text{TSRE}(N, \lambda)$ where we explicitly assume open boundary conditions; we shall, however, also consider a ring graph with periodic boundary conditions, in which case we shall stress this separately. In particular, we shall be interested in the zero-temperature (ground state) properties of $\text{TSRE}(N, \lambda)$. We note that due to the large co-dimension of bond strength space, the standard perturbative renormalization group method of decimating the strongest bonds [21] would not work and we have to, at this point rely on a brute numerical investigation.

Still, it turns out that most zero-temperature properties of $\text{TSRE}(N, \lambda)$ can be efficiently simulated using White's DMRG finite-size algorithm [19] by which spin chains of sizes up to $N = 80$ could, at present, be achieved⁴. One should not forget that all numerical estimates of *ensemble average* or expectation value of some physical quantity A , which will be designated as $\langle A \rangle$, require averaging over many, say \mathcal{N}_r , realizations from $\text{TSRE}(N, \lambda)$, such that the statistical error estimated as $\sigma_A \sim \sqrt{(\langle A^2 \rangle - \langle A \rangle^2)/\mathcal{N}_r}$ is sufficiently small. Due to the lack of translational invariance, the implementation of the DMRG is non-trivial and was only done for a chain with open boundaries.

We note that for $\lambda = 0$, any H as defined in (8), or even more generally in (1), commutes with the following anti-unitary *time-reversal* operation:

$$\hat{T} : \vec{\sigma}_j \rightarrow -\vec{\sigma}_j, \quad H|_{\lambda=0} \hat{T} = \hat{T} H|_{\lambda=0}, \quad (9)$$

so, for *odd* N , all eigenvalues of H have to be doubly degenerate (Kramers' degeneracy [15]). However, as this represents more a technical than conceptual problem for the ground state (or ground plane) properties of $\text{TSRE}(N, \lambda)$, we shall in the following restrict ourselves to the case of *even* N .

⁴ Using DMRG, one is able to efficiently calculate good approximations to a few (two) lowest-lying energy states by truncation to a small number (we took from 30 to 80) of the most entangled components. These are selected from the eigenvectors of the reduced density operator with the largest eigenvalues corresponding to a bipartite splitting. Note that these eigenvalues also directly determine the entanglement properties.

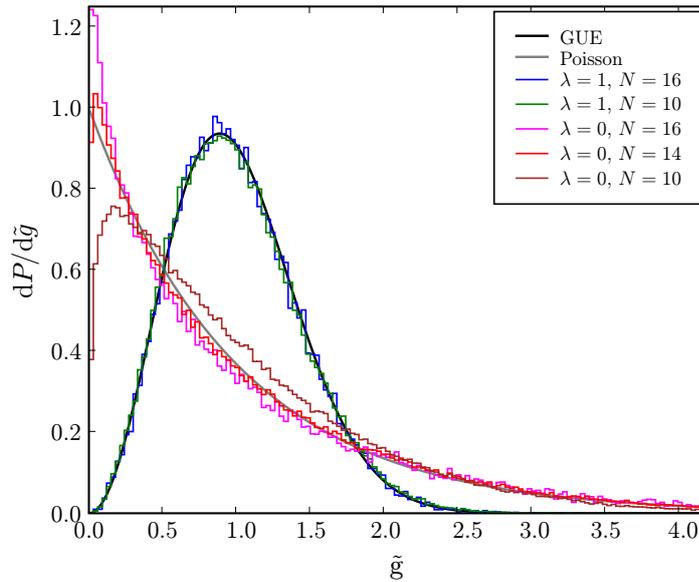


Figure 1. Distribution of normalized gap $\tilde{g} = g/\langle g \rangle$ for a few choices of even N . In the non- \hat{T} -invariant case, an agreement with GUE level spacing distribution is obtained, whereas in the \hat{T} -invariant case, $\lambda = 0$, the level repulsion gradually vanishes in the thermodynamic limit.

3.1. Distribution of the spectral gap

Let $|0\rangle$ and $|1\rangle$ represent the ground state and the first excited state of H , with eigenenergies E_0 and E_1 , respectively. It is well known that the crucial quantity that determines the rate of relaxation of zero-temperature quantum dynamics is the spectral gap $g = E_1 - E_0$.

To clarify the difference with respect to TRI, we plot the normalized gap distribution $dP/d\tilde{g}$ where $\tilde{g} = g/\langle g \rangle$ for both cases, together with the theoretical level spacing distributions for the Gaussian unitary ensemble (GUE) of random matrices [22] and for an uncorrelated Poissonian spectrum (figure 1). For the non-TRI case, we choose $\lambda = 1$ and observe, to our accuracy, good agreement with the GUE case for two chain sizes $N = 10$ and $N = 16$. Our results suggest that the GUE-like gap distribution, exhibiting level repulsion, also holds in the thermodynamic limit $N \rightarrow \infty$; numerical results for odd N give the same results. In the case of $\lambda = 0$ the level repulsion between $|0\rangle$ and $|1\rangle$ gradually vanishes as we approach the thermodynamic limit, although no conclusive statement can be made about the limiting distribution. For odd N , the ground state is degenerate for $\lambda = 0$ and the present analysis does not apply.

3.2. Size scaling of the spectral gap

Being interested in the thermodynamic limit, it is an important issue to understand how $g(N)$ scales with N . The theory of quantum criticality [16] states that g remains finite in the thermodynamic limit for *non-critical systems*, and rapidly converges to zero, as $N \rightarrow \infty$ for *critical systems*.

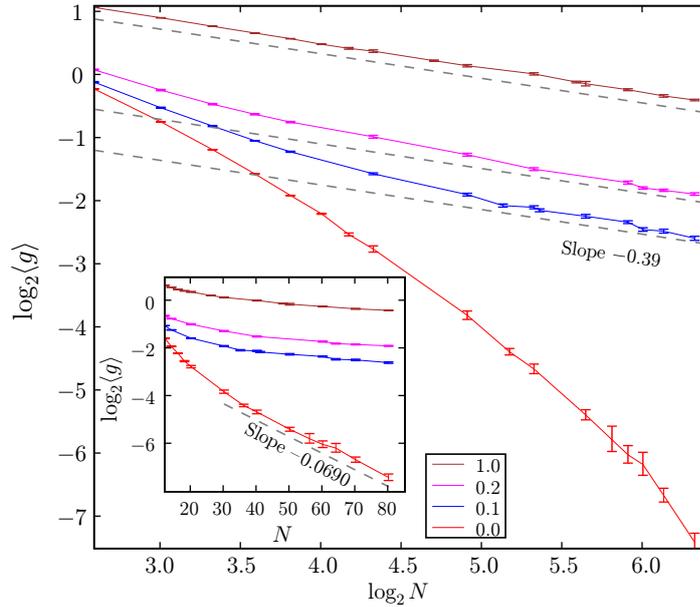


Figure 2. Spectral gap scaling with the system size for different values of parameter λ (see legend). Note an asymptotic scaling $\propto N^{-0.39}$, unless $\lambda = 0$ where faster than power law decay of a gap is observed, perhaps asymptotically exponential (see the inset for a semi-logarithmic scale).

In figure 2, we plot the ensemble-averaged spectral gap $\langle g \rangle$ versus N for different values of the field strength λ . We find a clear indication that in the non-TRI case the spectral gap exhibits universal asymptotic power law scaling

$$\langle g \rangle \sim N^{-\eta}, \quad \text{with } \eta \approx 0.39 \pm 0.01, \quad (10)$$

whereas in the TRI case, $\lambda = 0$, the asymptotic decay of the gap is *faster than a power law*, perhaps exponential $\langle g \rangle \sim \exp(-\xi N)$, with $\xi \approx 0.07 \pm 0.02$. According to the standard theory [16], both cases, $\lambda \neq 0$ and $\lambda = 0$, should be classified as quantum critical; however, as we shall see later, the case of slow power-law decaying average gap (10) has many of the features of non-critical systems, such as finite correlation length and finite (saturated) entanglement entropy. Therefore we shall, at least for the purposes of the present paper, name the case $\lambda \neq 0$ as *random non-critical* (RNC) and the case $\lambda = 0$ as *random critical* (RC).

We note that the results for odd and even N are in agreement in the RNC case. Also results for the case with periodic boundary conditions up to $N = 20$ show no significant difference from the results in figure 2 for any λ .

3.3. Size scaling of the ground state entanglement entropy

The second characteristics of quantum-phase transitions we choose to investigate in the TSRE(N, λ) is the entanglement entropy of a symmetric bipartition of the chain

$$S(N, \lambda) = -\langle \text{tr}_{\{N/2+1, \dots, N\}} \left[\left(\text{tr}_{\{1, \dots, N/2\}} |0\rangle \langle 0| \right) \log_2 \left(\text{tr}_{\{1, \dots, N/2\}} |0\rangle \langle 0| \right) \right] \rangle, \quad (11)$$

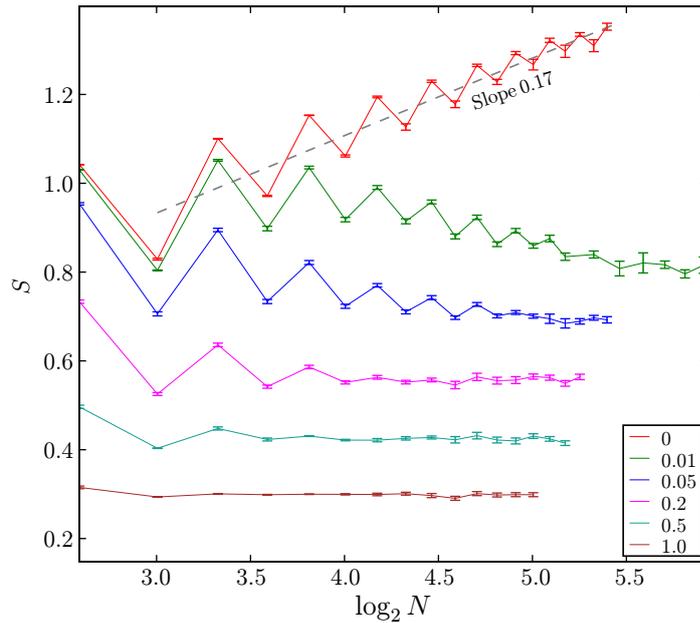


Figure 3. Entanglement entropy versus chain length N for different values of parameter λ (see the legend). Logarithmic divergence with an estimated asymptotic slope $S \sim 0.17 \log_2 N$ for the critical case $\lambda = 0$ is indicated with a dashed line.

which measures the entanglement in the ground state between two equal halves of the chain. It has been suggested in non-random systems [18] that for critical cases, $S \propto \log_2 N$, whereas in non-critical cases, S saturates in the thermodynamic limit.

Indeed, as shown in figure 3, we find for the TSRE(N, λ) that S saturates to a finite constant $S_\infty(\lambda) = \lim_{N \rightarrow \infty} S(N, \lambda)$ for the RNC case $\lambda \neq 0$, while in the RC case it grows logarithmically

$$S(N, \lambda = 0) \approx c \log_2 N + c', \quad \text{with } c \approx 0.17 \pm 0.02. \quad (12)$$

We also note an interesting even–odd- $N/2$ effect which slowly diminishes as we approach the thermodynamic limit. As pointed out in [23], such an effect is induced by open boundary conditions. For periodic boundary conditions, the entanglement entropy for the RNC case is twice as large as in the TSRE(N, λ) with open boundaries. This further confirms the conjecture that only short-range correlations around the boundary between the two halves contribute to the entanglement.

We note that our result is essentially different from results for other models, which can be obtained by perturbative real space renormalization group [21], for example, for the disordered critical Heisenberg chain [24], where $c = (\ln 2)/3$ and is in general model dependent [25].

The fact that the entanglement is reduced in the RNC case with local disorder can be explained by chaotic behavior [26] signaled by the level repulsion in the gap distribution. A similar effect can be observed in localization properties where hopping of excitations induced by inter-particle interactions is diminished by introducing local disorder [27, 28]. This effect of increased localization is useful for successful quantum computing and has important consequences for transport properties such as conductivity [29].

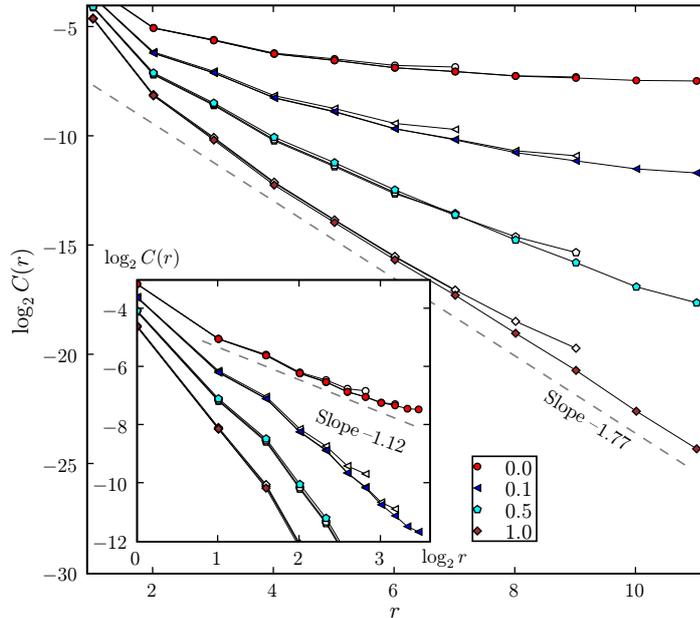


Figure 4. Ensemble fluctuations of the ground state spin–spin correlation function $C(r)$ versus distance r , for different values of the parameter λ (indicated in the legend). Open symbols indicate the results for chain lengths $L = 16$ and $L = 20$, whereas closed symbols stand for $L = 24$.

3.4. Correlation functions

The most direct probe of criticality is perhaps to investigate long-range order and (space) correlation functions. In order to do this, we compute the ensemble-averaged fluctuation of the spin–spin correlation function between two vertices

$$C(j, k) = \langle |\langle 0 | \sigma_j^\alpha \sigma_k^\beta | 0 \rangle - \langle 0 | \sigma_j^\alpha | 0 \rangle \langle 0 | \sigma_k^\beta | 0 \rangle|^2 \rangle. \quad (13)$$

Note that we have to consider average fluctuations of the spin–spin correlation function instead of the correlation function itself, since the latter have to vanish due to the local gauge invariance properties of the TSRE. Because of the local invariance of the TSRE, it is enough to consider a single type of correlation function, as the rhs of (13) does not depend on indices α, β if $j \neq k$; in fact in numerical computations we average over α, β in order to improve statistics. We consider the Hamiltonian (8) with periodic boundary conditions. This allows us to average the fluctuation of the correlation over the chain and hence $C(r) = \frac{1}{N} \sum_i C(i, i+r)$. We expect that the results would be qualitatively the same for the model with open boundaries and sufficiently large N , but we obtain better statistics in this way.

Figure 4 shows the averaged correlation function fluctuation $C(r)$ for a few choices of the control parameter λ and of the chain lengths $N = 16, 20$ and 24 . The results for chains of different lengths coincide for small distances r , whereas for larger r , finite-size effects are noticeable. For sufficiently large chains, it can be conjectured that the fluctuations (or effective correlations) asymptotically decay in the RNC case $C(r) \asymp C_0 2^{-r/\xi}$ with a finite correlation length ξ , whereas the decay for the RC case is slower than exponential, perhaps a power law, which indicates long-range order $\xi = \infty$.

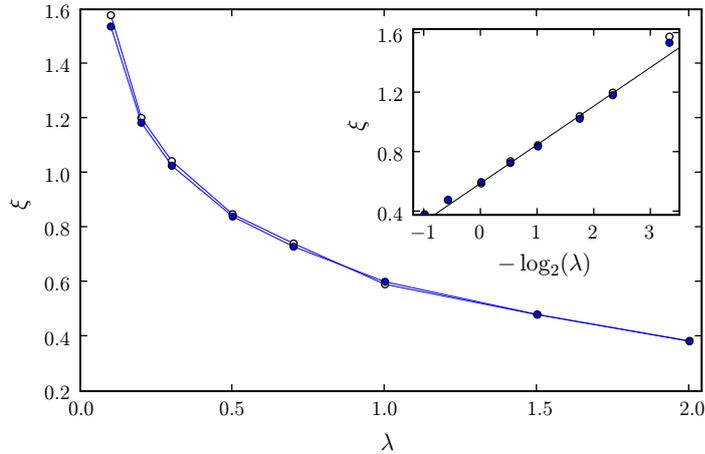


Figure 5. The correlation length ξ as a function of the control parameter λ . Open and full symbols designate correlation length obtained from the best of $C(r)$ with $c \exp(-r/\xi)$ on sets $r \in \{4, \dots, 8\}$ and $r \in \{5, \dots, 7\}$, respectively, all for $N = 24$. The inset demonstrates $\propto -\log_2 \lambda$ scaling. Please note that the last point at $\lambda = 0.1$ is rather inaccurate due to insufficiently large system size N , so it is not used in a linear fit (full line) in the inset, for which other points with $\lambda \leq 1$ have been used.

3.5. Correlation length and the entanglement entropy saturation value

In figure 4, we observe that the degree of localization depends on the control parameter λ and the correlation function $C(r)$ decays on larger scales as we approach the critical point $\lambda = 0$, which results in a larger correlation length ξ . Eventually, the correlation length becomes infinite at the critical point $\lambda = 0$. In figure 5, we show the dependence of the correlation length ξ on the control parameter λ as obtained by exponential fit of $C(r)$ for a finite size $N = 24$. Unlike conventional phase transition as in e.g. [17], the correlation length seems to diverge logarithmically as $\xi(\lambda) \sim -\xi_0 \log_2 \lambda + \text{const.}$ with $\xi_0 = 0.26$ (indicated in the inset of figure 5), even though algebraic scaling cannot be entirely excluded with the numerical data that are available at present.

A large correlation length has a strong effect on entanglement. Long-range correlations demand longer chains for the entanglement entropy to saturate, whereas the saturation value itself also grows when the critical point is approached. In figure 6, we plot the entanglement entropy saturation value for various values of parameter λ and observe similar behavior as for the correlation length. However, unlike the correlation length, the quantity that diverges logarithmically when $\lambda \rightarrow 0$ is not the entanglement entropy S but its exponential. In fact, numerical data for small λ show good agreement with $2^{kS} \propto -\log_2 \lambda + \text{const.}$, where $k = 4.0$ (indicated in the inset of figure 6). Note that 2^S is, to a good approximation, proportional to the effective rank, or the Schmidt number of the ground state χ^ϵ [18, 30], which denotes the number of eigenvalues of the reduced density matrix needed to describe the state of the system up to an error ϵ . In fact, the effective rank χ^ϵ , rather than the entanglement entropy, is the decisive indicator of simulability by the DMRG method [31] and, we believe, also a relevant quantity in the description of a quantum-phase transition.

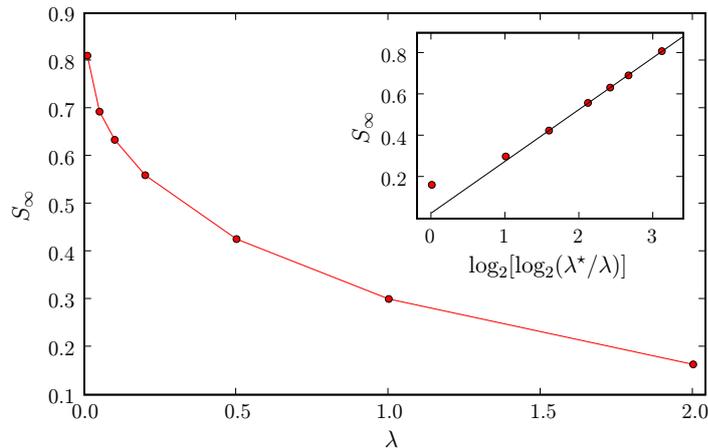


Figure 6. Saturated entanglement entropy S_∞ as a function of the control parameter λ . The inset demonstrates $\propto \log_2 \log_2(\lambda^*/\lambda)$ scaling, where $\lambda^* = 4$, which is indicated by a straight line obtained from a best fit to points with $\lambda < 1$.

4. Conclusions

In the present paper, we have defined a two-body random matrix ensemble of independent spin Hamiltonians which are invariant under local $SO(3)$ transformations and described them in a framework of undirected graphs. As the simplest example, we have studied a chain with nearest-neighbor interactions in a random external field and observed a non-conventional phase transition when the external field is switched off. The system is always critical in conventional terminology as it has a vanishing gap in the thermodynamic limit in all the cases studied. Yet we have shown that, in the presence of a random external field breaking TRI, the locally disordered system has many of the properties of non-critical systems such as finite correlation length and finite bipartite entanglement entropy in the thermodynamic limit, whereas the gap decay obeys a universal power law dependence. The transition towards the critical point with vanishing of the external field exhibits logarithmic divergence for the correlation length and the effective rank of the ground state. We have no explanation for the logarithmic behavior in the quantum-phase transition.

The model proposed is much richer than the example discussed. Thus we expect that higher connectivity of the graph will yield very different results, but even an exploration of high-temperature behavior for the chain seems very worthwhile. In view of the large structural invariance group of the ensemble in the case of site-independent average coupling and external fields, we hope that some analytic results can be obtained for this ensemble.

Acknowledgments

We acknowledge support from the Slovenian Research Agency, program P1-0044 and grant J1-7437, from the CONACyT under grant 57334 and from the UNAM-PAPIIT under grant IN112507. IP and TP thank THS and the CIC (Cuernavaca, Mexico) for hospitality.

References

- [1] Wigner E P 1951 *Ann. Math.* **53** 36
- [2] Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S M 1981 *Rev. Mod. Phys.* **53** 385
- [3] Guhr T, Müller-Groeling A and Weidenmüller H A 1998 *Phys. Rep.* **299** 189
- [4] Gorin T and Seligman T H 2002 *J. Quantum Opt.* B **4** S386
- [5] Gorin T, Prosen T and Seligman T H 2004 *New J. Phys.* **6** 20
- [6] Frahm K M, Fleckinger R and Shepelyansky D L 2004 *Eur. Phys. J. D* **29** 139
- [7] Pineda C and Seligman T H 2007 *Phys. Rev. A* **75** 012106
- [8] Pineda C, Gorin T and Seligman T H 2007 *New J. Phys.* **9** 106
- [9] Pineda C and Seligman T H 2008 *ELAF XXXVIII (2007) Proc. Am. Inst. Phys.* at press
- [10] Benet L and Weidenmüller H A 2003 *J. Phys. A: Math. Gen.* **36** 3569
- [11] Flores J, Horoi M, Müller M and Seligman T H 2000 *Phys. Rev. E* **63** 026204
- [12] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
- [13] Zutic I 2004 *Rev. Mod. Phys.* **76** 323
- [14] Balian R 1958 *Nuovo Cimento B* **57** 183
- [15] Haake F 2001 *Quantum Signatures of Chaos* (Berlin: Springer)
- [16] Sachdev S 1999 *Quantum Phase Transitions* (Cambridge: Cambridge University Press)
- [17] Osterloh A, Amico L, Falci G and Fazio R 2002 *Nature* **416** 608
- [18] Vidal G, Latorre J I, Rico E and Kitaev A 2003 *Phys. Rev. Lett.* **90** 227902
- [19] White S R 1992 *Phys. Rev. Lett.* **69** 2863
White S R 1993 *Phys. Rev. B* **48** 10345
- [20] Horodecki R and Horodecki M 1996 *Phys. Rev. A* **54** 1838
- [21] Dasgupta C and Ma S 1980 *Phys. Rev. B* **22** 1305
Fisher D S 1994 *Phys. Rev. B* **50** 3799
- [22] Mehta M L 1991 *Random Matrices* (New York: Academic)
- [23] Laflorencie N, Sørensen E S, Chang M-S and Affleck I 2007 *Phys. Rev. Lett.* **96** 100603
- [24] Refael G and Moore J E 2004 *Phys. Rev. Lett.* **93** 260602
- [25] Santachiara R 2006 *J. Stat. Mech.* **L06002**
- [26] Santos L F, Rigolin G and Escobar C O 2004 *Phys. Rev. A* **69** 042304
Santos L F 2004 *J. Phys. A: Math. Gen.* **37** 4723
- [27] Dykman M I, Izrailev F M, Santos L F and Shapiro M 2004 *Preprint cond-mat/0401201*
- [28] Giraud O, Martin J and Georgeot B 2007 *Phys. Rev. A* **76** 042333
- [29] Basko D M, Aleiner I L and Altshuler B L 2006 *Ann. Phys.* **321** 1126 (*Preprint cond-mat/0602510*)
- [30] Vidal G 2003 *Phys. Rev. Lett.* **91** 147902
Vidal G 2004 *Phys. Rev. Lett.* **93** 040502
White S R and Feiguin A E 2004 *Phys. Rev. Lett.* **93** 076401
Daley A J, Kollath C, Schollwöck U and Vidal G 2004 *J. Stat. Mech.* **P04005**
- [31] Schuch N, Wolf M M, Verstraete F and Cirac J I 2007 *Preprint 0705.0292*