

Quantum freeze of fidelity decay for a class of integrable dynamics

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Abstract. We discuss quantum fidelity decay of classically regular dynamics, in particular for an important special case of a vanishing time-averaged perturbation operator, i.e. vanishing expectation values of the perturbation in the eigenbasis of unperturbed dynamics. A complete semiclassical picture of this situation is derived in which we show that the quantum fidelity of *individual coherent initial states* exhibits three different regimes in time: (i) first it follows the corresponding classical fidelity up to time $t_1 \sim \hbar^{-1/2}$, (ii) then it *freezes* on a plateau of constant value, (iii) and after a timescale $t_2 \sim \min\{\hbar^{1/2}\delta^{-2}, \hbar^{-1/2}\delta^{-1}\}$ it exhibits fast ballistic decay as $\exp(-\text{constant} \times \delta^4 t^2 / \hbar)$ where δ is a strength of perturbation. All the constants are computed in terms of classical dynamics for sufficiently small effective value \hbar of the Planck constant. A similar picture is worked out also for general initial states, and specifically for random initial states, where $t_1 \sim 1$, and $t_2 \sim \delta^{-1}$. This prolonged stability of quantum dynamics in the case of a vanishing time-averaged perturbation could prove to be useful in designing quantum devices. Theoretical results are verified by numerical experiments on the quantized integrable kicked top.

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1. Introduction

The squared modulus of the overlap between a pair of time evolving quantum states propagated by two slightly different Hamiltonians, known as the fidelity or the quantum Loschmidt echo, has recently attracted a lot of attention [1]–[7]. In addition to numerous numerical simulations, several theoretical results have been proposed for describing the fidelity decay in relation to the nature of the (corresponding classical) dynamics. Jalabert and Pastawski [2] have related fidelity decay for coherent initial states at very short times, namely below or around the Ehrenfest time $\propto \log \hbar$, to the classical phase space stretching rate as characterized by the Lyapunov exponents. In more general situations, and in particular for longer timescales, fidelity decay has been related to the integrated time correlation function of the perturbation through a kind of fluctuation-dissipation relationship [4]–[6].

In a recent paper [6] we have developed a general theory of fidelity decay based on a semiclassical treatment of this fluctuation-dissipation formula. It turns out that if the corresponding classical dynamics is fully chaotic then the decay of fidelity is, after a short $\sim \log \hbar$ (Ehrenfest) timescale, independent of the structure of the initial state in accordance with the quantum ergodicity. In contrast, if the corresponding classical dynamics is regular then the long time asymptotics sensitively depend on the structure of the initial state and range from a *Gaussian* fidelity decay for *coherent initial states* to a *power law* fidelity decay for *random initial states*. For regular classical dynamics the theory [6] predicts *faster* decay of fidelity on a short timescale, $\propto \delta^{-1}$ (δ = strength of the perturbation), as the time correlation function of the perturbation observable does not decay, *compared* to the chaotic classical dynamics where the

decay timescale is longer, $\propto \delta^{-2}$, and is longer the faster the decay of correlations that we have. However, the fast ballistic decay of fidelity in the case of regular classical dynamics described by the theory [6] does not happen in one special but important case, namely when the time average of the perturbation (i.e. the observable which perturbs the Hamiltonian) vanishes. Classically, this means that the perturbation does not change the frequencies of the invariant (KAM) tori in the leading order in δ , at least in the phase space region of interest.

Such a case of regular classical dynamics with vanishing time-averaged perturbation is the subject of the present paper. Though this is not a generic case for a sufficiently large class of perturbations, it may emerge naturally if the system and the perturbation possess appropriate discrete or continuous symmetries. We will discuss general initial states, and specifically also coherent and random initial states. We find a very surprising result, namely that the quantum fidelity, after decaying for a short time (e.g. following the classical fidelity [6, 8, 9] for coherent initial states), *freezes* on a plateau of constant value. This is purely a quantum effect and has no analogue in the classical fidelity. The relative time span of the plateau is of the order of the inverse perturbation strength $1/\delta$ and can be made arbitrarily large for small perturbations. However, for long times after the plateau ends, the fidelity displays a ballistic decay with the characteristic timescale $\propto 1/\delta^2$, e.g. Gaussian for coherent initial states and power law t^{-d} for random initial states where d is the number of freedoms. This ballistic decay can be explained semiclassically due to perturbative changes of the frequencies of invariant tori in the second order in δ . For coherent initial states in one dimension we find and explain another quite surprising general phenomenon, which we call ‘the echo resonance’ where the fidelity displays sudden and significant revivals which can, under certain conditions, come back even to value 1. This happens at particular values of times, which depend on the derivative of the classical frequency with respect to the canonical action and do not depend on δ .

Using the formalism of action–angle variables and its semiclassical quantization we derive explicit semiclassical formulae, in the leading order in \hbar , for the fidelity in all the regimes. Our results are demonstrated with high precision by numerical experiments using a regular quantum top which is perturbed by periodic kicking. The quantum saturation of fidelity, which is a central result of this paper, may also be of some practical importance as it provides a mechanism for stabilizing the regular quantum dynamics.

In section 2 we define the basic quantities and study the general properties of the so-called *echo operator* whose expectation value gives the fidelity. We propose a useful asymptotic ansatz for the echo operator, which is used later in section 3, in combination with the semiclassical action–angle dynamics, to derive explicit general results on the echo operator and fidelity and to identify different regimes. In section 4 we define a numerical model on which the results for two specific classes of initial states, namely coherent and random states, are later quantitatively validated in sections 5 and 6, respectively. In section 7 we discuss the general picture and summarize the results.

2. Quantum mechanics of the echo operator

Let H_0 and $H_\delta = H_0 + H'\delta$ denote the *unperturbed* and the *perturbed* Hamiltonian, respectively. In order to cover the even more general case of periodically time dependent (e.g. kicked) systems, say of period τ , $H_\delta(\tau' + \tau) = H_\delta(\tau')$, we utilize our formalism in terms of the Floquet map $U_\delta = [\hat{T} \exp(i \int_0^\tau d\tau' H_\delta(\tau')/\hbar)]^\dagger$, where \hat{T} denotes a *left-to-right* time-ordered product. The dynamics is now generated by a discrete group U_δ^t , $t \in \mathbb{Z}$, where an autonomous continuous

time flow is approached in the limit $\tau \rightarrow 0$. It seems convenient to postulate a slightly different but completely general form of a *small* perturbation

$$U_\delta = U_0 \exp(-iV\tau\delta/\hbar) \quad (1)$$

generated by a Hermitian operator V which in the leading order matches H' , $V = H' + \mathcal{O}(\tau\delta)$. We note that all the results in the paper can be trivially translated to the continuous time case by making the substitutions $t\tau \rightarrow T$, $\tau \sum_{t'=0}^{t-1} \rightarrow \int_0^T dt'$.

Starting from the same initial state $|\psi\rangle$, the *fidelity* or the *Loschmidt echo* $F(t)$ is defined as the squared modulus of the overlap between $U_0^t|\psi\rangle$ and $U_\delta^t|\psi\rangle$, namely

$$F(t) = |f(t)|^2, \quad f(t) = \langle \psi | M_\delta(t) | \psi \rangle, \quad (2)$$

where $f(t)$ is called the *fidelity amplitude* and

$$M_\delta(t) = U_\delta^{-t} U_0^t \quad (3)$$

is the *echo operator*. Equivalently, $M_\delta(t)$ is the time-ordered propagator generated by the perturbation $V_t = U_0^{-t} V U_0^t$ in the interaction picture [4, 6]:

$$M_\delta(t) = \prod_{t'=0}^{t-1} \exp\left(i\frac{\tau\delta}{\hbar} V_{t'}\right) = \hat{T} \exp\left(i\frac{\tau\delta}{\hbar} \sum_{t'=0}^{t-1} V_{t'}\right). \quad (4)$$

The essential results on the behaviour of fidelity [6] are then derived from the combination of perturbative and semiclassical considerations of the formula (4). For example, the essential physics is contained in a linear response approximation which is obtained by expanding (4) to second order in δ :

$$F(t) = 1 - \frac{\tau^2\delta^2}{\hbar^2} \sum_{t',t''=0}^{t-1} C(t',t'') + \dots, \quad (5)$$

$$C(t',t'') = \langle V_{t'} V_{t''} \rangle - \langle V_{t'} \rangle \langle V_{t''} \rangle, \quad (6)$$

where $\langle \bullet \rangle := \langle \psi | \bullet | \psi \rangle$ is the expectation value in the initial state $|\psi\rangle$. Thus stronger decay of correlations qualitatively enhances the stability of quantum motion [4]–[6]. It is useful to rewrite the double sum on the RHS of linear response formula (5) in terms of the uncertainty of the *integrated perturbation operator*

$$\Sigma_t = \tau \sum_{t'=0}^{t-1} V_{t'}, \quad (7)$$

that is, as

$$F(t) = 1 - \frac{\delta^2}{\hbar^2} \{ \langle \Sigma_t^2 \rangle - \langle \Sigma_t \rangle^2 \} + \mathcal{O}(\delta^4). \quad (8)$$

Here we take a slightly different route and apply the Baker–Campbell–Hausdorff (BCH) expansion $e^A e^B = \exp(A + B + (1/2)[A, B] + \dots)$ to the echo operator (4)

$$\begin{aligned} M_\delta(t) &= \exp\left\{ i\frac{\tau\delta}{\hbar} \sum_{t'=0}^{t-1} V_{t'} - \frac{\tau^2\delta^2}{2\hbar^2} \sum_{t'=0}^{t-1} \sum_{t''=t'}^{t-1} [V_{t'}, V_{t''}] + \dots \right\} \\ &= \exp\left\{ \frac{i}{\hbar} \left(\Sigma_t \delta + \frac{1}{2} \Gamma_t \delta^2 + \dots \right) \right\} \end{aligned} \quad (9)$$

where $[A, B] := AB - BA$, introducing another operator valued series

$$\Gamma_t = \frac{i\tau^2}{\hbar} \sum_{t'=0}^{t-1} \sum_{t''=t'}^{t-1} [V_{t'}, V_{t''}]. \quad (10)$$

Note that for systems with a well defined classical limit the operator Γ_t corresponds to an \hbar independent classical observable as $(-i/\hbar)[\bullet, \bullet]$ corresponds to the classical Poisson bracket. In the ergodic and mixing case, of say classically strongly chaotic dynamics, straightforward expansion of the exponential (4) gives the Fermi-golden-rule [7] exponential decay $F(t) = \exp(-\kappa t)$ [4, 6] where the argument κt is precisely the double sum of the correlation function on the RHS of (5) for sufficiently long times. However, in the opposite case of classically regular (integrable) dynamics, on which we focus in the following sections of this paper, the BCH form (9) will turn out to be particularly useful.

Let us first generally discuss the expression (9) from the point of view of exact unitary quantum dynamics. For a typical observable V , one can define a non-trivial time-average operator

$$\bar{V} = \lim_{t \rightarrow \infty} \frac{\Sigma_t}{t\tau} = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{t'=0}^{t-1} V_{t'}, \quad (11)$$

which is by construction an invariant of motion, $[U_0, \bar{V}] = 0$. In a generic case of a *non-degenerate spectrum* of U_0 , the time average is simply the diagonal part in the eigenbasis $|n\rangle$ of the unperturbed evolution, $U_0|n\rangle = e^{-i\varphi_n}|n\rangle$, namely

$$\bar{V} = \sum_n V_{nn}|n\rangle\langle n|, \quad (12)$$

where $V_{nm} := \langle n|V|m\rangle$. In general we split the perturbation into a sum of a diagonal and a residual part

$$V = \bar{V} + V_{\text{res}}. \quad (13)$$

We say that the observable V is *residual* if $V = V_{\text{res}}$. This corresponds to ergodicity of this specific observable, namely $\bar{V} = 0$, meaning that V has zero diagonal elements, and this is clearly a special (non-generic) situation¹. In this paper we discuss integrable dynamics and the class of residual perturbations. For non-degenerate eigenphases $\{\varphi_n\}$ the matrix elements of the second order term (10) in the BCH expansion can be straightforwardly calculated in the leading order in t as

$$\begin{aligned} \frac{\langle n|\Gamma_t|n\rangle}{t\tau} &= \frac{\tau}{\hbar} \sum_{k \neq n} |V_{nk}|^2 \cot \left[\frac{1}{2}(\varphi_k - \varphi_n) \right] + \mathcal{O}(t^{-1}), \\ \frac{\langle n|\Gamma_t|m\rangle}{t\tau} &= \frac{\tau}{\hbar} (V_{nn} - V_{mm}) V_{nm} \frac{e^{-i\frac{1}{2}(\varphi_n - \varphi_m)} + e^{-i(\varphi_n - \varphi_m)(\frac{1}{2}-t)}}{2 \sin[\frac{1}{2}(\varphi_m - \varphi_n)]} + \mathcal{O}(t^{-1}), \quad n \neq m. \end{aligned} \quad (14)$$

Hence we see that, provided that the perturbation is residual, $\bar{V} = 0$, the limit of *doubly averaged perturbation* defined as

¹ We can always choose the perturbation V to be *traceless*, since subtracting a constant, $V \rightarrow V - \mathbb{1} \text{tr} V / \text{tr} \mathbb{1}$, only changes the phase of the amplitude f and does not affect the fidelity $F = |f|^2$.

$$\bar{V} = \lim_{t \rightarrow \infty} \frac{\Gamma_t}{t\tau} = \frac{i\tau}{\hbar} \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{t'=0}^{t-1} \sum_{t''=t'}^{t-1} [V_{t'}, V_{t''}] \quad (15)$$

exists and is *diagonal* in the eigenbasis of U_0 :

$$\bar{V} = \sum_n \bar{V}_{nn} |n\rangle \langle n|, \quad \bar{V}_{nn} = \frac{\tau}{\hbar} \sum_k^{k \neq n} |V_{nk}|^2 \cot \left[\frac{1}{2} (\varphi_k - \varphi_n) \right]. \quad (16)$$

Note that \bar{V} is again an invariant of motion, $[U_0, \bar{V}] = 0$, and that, unlike for the time average \bar{V} , its trace always vanishes, $\text{tr} \bar{V} = 0$.

In the *generic* case, $\bar{V} \neq 0$ is a non-trivial operator. For sufficiently small perturbation the second term in the exponent of RHS of (9) can always be neglected, since its arbitrary (finite) norm grows as $t\delta^2$ following from $|\langle n | \Gamma_t | m \rangle| < \text{constant} \times t$ (see equation (14)), in comparison to the first term whose norm grows as $t\delta$. For sufficiently long times, i.e. longer than the effective convergence time of the limit (11), we can write $\Sigma_t \rightarrow t\tau \bar{V}$ so the echo operator can be written as $M_\delta(t) = \exp(i\bar{V}t\tau\delta/\hbar)$ from which useful semiclassical expressions for initial states of different types were derived [6], all showing fidelity decay on an effective timescale that is $\propto \delta^{-1}$. In the specific case of *residual* perturbation, $\bar{V} = 0$, the norm of the first term in the exponential on RHS of (9) does not grow in time, as we shall discuss in the next section, so the second term will dominate for sufficiently long times.

Although residual perturbations are not generic in the entire set of physically admissible perturbations V , they may nevertheless be of particular interest in cases where one is allowed to shift the entire diagonal part of the matrix V_{nm} to the unperturbed Hamiltonian matrix, which is diagonal by definition. Also, it is easy to imagine practically or experimentally important situations where vanishing of the diagonal part, $V_{nn} \equiv 0$, is required by the *symmetry*. For example, it is obvious that having a unitary symmetry operation R , $R^\dagger R = 1$, commuting with the unperturbed evolution, $[R, U_0] = 0$, and the perturbation V which has a negative ‘parity’ with respect to the symmetry operation, $R^\dagger V R = -V$, is sufficient to give $V_{nn} = 0$.

As the case of generic perturbations has been treated in detail in previous publications [4, 6], we shall from now on entirely concentrate on the residual case $\bar{V} = 0$, unless explicitly stated otherwise. In this case we have found the following uniform approximation of the echo operator:

$$M_\delta(t) = \exp \left\{ \frac{i}{\hbar} \left(\Sigma_t \delta + \frac{1}{2} \bar{V} t \tau \delta^2 \right) \right\}, \quad (17)$$

which is accurate, for sufficiently small δ , up to long times at least of the order of $1/\delta^2$. This is a consequence of the fact that for $\bar{V} = 0$ the third order term in BCH expansion (9) again grows only linearly in time, $\sim t\delta^3$, and that the fourth order term cannot grow faster than $\sim t^2\delta^4$. The rest of the paper will be dedicated to the semiclassical exploration of formula (17).

3. Semiclassical asymptotics

Since the classical mechanics is assumed to be completely integrable (at least locally, by the KAM theorem, in the phase space part of interest) we can write the classical limit $v(\mathbf{j}, \boldsymbol{\theta})$ of the perturbation operator V in canonical action–angle variables $\{j_k, \theta_k, k = 1, \dots, d\}$ with d degrees of freedom as the Fourier series in d dimensions

$$v(\mathbf{j}, \boldsymbol{\theta}) = \sum_{\mathbf{m} \in \mathbb{Z}^d} v_{\mathbf{m}}(\mathbf{j}) e^{i\mathbf{m} \cdot \boldsymbol{\theta}}. \quad (18)$$

We shall throughout the paper use lower/upper case letters to denote the corresponding classical/quantum observables. Note that the classical limit of the unperturbed Hamiltonian H_0 can be written as a function $h_0(\mathbf{j})$ of the canonical actions only, yielding the well known quasi-periodic solution of Hamilton's equations

$$\begin{aligned} \mathbf{j}_t &= \mathbf{j}, \\ \boldsymbol{\theta}_t &= \boldsymbol{\theta} + \boldsymbol{\omega}(\mathbf{j})t \pmod{2\pi}, \end{aligned} \quad (19)$$

with the dimensionless frequency vector

$$\boldsymbol{\omega}(\mathbf{j}) := \tau \frac{\partial h_0(\mathbf{j})}{\partial \mathbf{j}}. \quad (20)$$

The classical limit of the time-averaged perturbation \bar{V} is $\bar{v} = v_0(\mathbf{j})$ which is here assumed to vanish: $v_0(\mathbf{j}) \equiv 0$.

In quantum mechanics, one quantizes the action–angle variables using the famous EBK procedure [13] where one defines the action (momentum) operators \mathbf{J} and angle operators $\exp(i\mathbf{m} \cdot \boldsymbol{\Theta})$ satisfying the canonical commutation relations

$$[J_k, \exp(i\mathbf{m} \cdot \boldsymbol{\Theta})] = \hbar m_k \exp(i\mathbf{m} \cdot \boldsymbol{\Theta}), \quad k = 1, \dots, d. \quad (21)$$

As the action operators are mutually commuting, they have a common eigenbasis $|\mathbf{n}\rangle$ labelled by a d -tuple of quantum numbers $\mathbf{n} = (n_1, \dots, n_d)$,

$$\mathbf{J}|\mathbf{n}\rangle = \hbar(\mathbf{n} + \boldsymbol{\alpha})|\mathbf{n}\rangle \quad (22)$$

where $0 \leq \alpha_k \leq 1$ are the Maslov indices which are irrelevant for the leading order semiclassical approximation employed in this paper. It follows from equation (21) that the angle operators act as shifts:

$$\exp(i\mathbf{m} \cdot \boldsymbol{\Theta})|\mathbf{n}\rangle = |\mathbf{n} + \mathbf{m}\rangle. \quad (23)$$

The Heisenberg equations of motion can be trivially solved, disregarding the operator ordering problem in the leading semiclassical order:

$$\begin{aligned} \mathbf{J}_t &= e^{iH_0\tau t/\hbar} \mathbf{J} e^{-iH_0\tau t/\hbar} = \mathbf{J}, \\ e^{i\mathbf{m} \cdot \boldsymbol{\Theta}_t} &= e^{iH_0\tau t/\hbar} e^{i\mathbf{m} \cdot \boldsymbol{\Theta}} e^{-iH_0\tau t/\hbar} \cong e^{i\mathbf{m} \cdot \boldsymbol{\omega}(\mathbf{J})t} e^{i\mathbf{m} \cdot \boldsymbol{\Theta}}, \end{aligned} \quad (24)$$

in terms of the frequency operator $\boldsymbol{\omega}(\mathbf{J})$. Throughout the whole paper we use the symbol \cong for ‘semiclassically equal’, i.e. asymptotically equal in the leading order in \hbar . Similarly, time evolution of the perturbation observable is obtained in the leading order by replacement of classical with quantal action–angle variables in the expression (18)

$$V_t = e^{iH_0\tau t/\hbar} V e^{-iH_0\tau t/\hbar} \cong \sum_{\mathbf{m} \neq \mathbf{0}} v_{\mathbf{m}}(\mathbf{J}) e^{i\mathbf{m} \cdot \boldsymbol{\omega}(\mathbf{J})t} e^{i\mathbf{m} \cdot \boldsymbol{\Theta}}. \quad (25)$$

Now we are ready to write out the semiclassical expressions for the two-term BCH expansion (17). The operator Σ_t (7) giving the first order BCH term can be computed as a trivial geometric series

$$\Sigma_t \cong \sum_{m \neq 0} \tilde{v}_m(\mathbf{J})(1 - e^{im \cdot \omega(\mathbf{J})t})e^{im \cdot \Theta} \quad (26)$$

yielding a quasi-periodic and hence bounded temporal behaviour (due to $v_0(\mathbf{j}) = 0$). Here we have introduced modified Fourier coefficients $\tilde{v}_m(\mathbf{j})$ of the perturbation

$$\tilde{v}_m(\mathbf{j}) = \frac{\tau v_m(\mathbf{j})}{1 - e^{im \cdot \omega(\mathbf{j})}} = ie^{-im \cdot \omega(\mathbf{j})/2} \frac{\tau v_m(\mathbf{j})}{2 \sin(\frac{1}{2}m \cdot \omega(\mathbf{j}))}. \quad (27)$$

As for the operator \bar{V} , or Γ_t , giving the second order BCH term, the calculation is more tedious. First, we plug the semiclassical dynamics (25) into the definition (15). Second, we compute the resulting commutators of the form

$$[v_{m'}(\mathbf{J})e^{im' \cdot \omega(\mathbf{J})t'} e^{im' \cdot \Theta}, v_{m''}(\mathbf{J})e^{im'' \cdot \omega(\mathbf{J})t''} e^{im'' \cdot \Theta}],$$

in the leading order in \hbar , by means of the Poisson brackets

$$i\hbar(\partial_\theta(v_m e^{im' \cdot (\theta + \omega t')}) \cdot \partial_j(v_{m''} e^{im'' \cdot (\theta + \omega t'')}) - \partial_j(v_{m'} e^{im' \cdot (\theta + \omega t')}) \cdot \partial_\theta(v_{m''} e^{im'' \cdot (\theta + \omega t'')}))$$

and replacement of variables \mathbf{j} , $e^{im \cdot \theta}$ by operators \mathbf{J} , $e^{im \cdot \Theta}$. Third, we drop the terms for which $m' + m'' \neq 0$, since these contain the shift operator $e^{i(m'+m'') \cdot \Theta}$ (equation (23)), giving off-diagonal matrix elements only, which we know should give vanishing overall contribution for a residual observable, see equations (14) and (16).² As a result we find the following semiclassical expression:

$$\begin{aligned} \bar{V} &\cong \bar{v}(\mathbf{J}), \\ \bar{v}(\mathbf{j}) &= -\frac{\tau}{2} \sum_{m \neq 0} m \cdot \partial_j \left\{ |v_m(\mathbf{j})|^2 \cot\left(\frac{1}{2}m \cdot \omega(\mathbf{j})\right) \right\}. \end{aligned} \quad (28)$$

We have derived the semiclassical expressions for both terms occurring in the echo operator (17), for the integrated perturbation Σ_t (26), and for the doubly averaged perturbation \bar{V} (28). However, we note that both semiclassical expressions (26), (28) are subject to a potential ‘small denominator’ problem which is closely related to the one in KAM theory. This well known problem of divergence of sums over the Fourier index \mathbf{m} can be avoided in a generic case. First, strict singularities at resonances $m \cdot \omega = 0 \pmod{2\pi}$, where frequencies ω are evaluated in the eigenstates $|\mathbf{n}\rangle$, happen with probability zero. Second, the *near resonances* give a finite total contribution if one assumes that the classical limit of the perturbation $v(\mathbf{j}, \theta)$ is sufficiently smooth, e.g. analytic in angles θ , such that the Fourier coefficients v_m fall off exponentially in $|\mathbf{m}|$. This will be assumed throughout the rest of this paper, whereas the cases of more singular perturbations call for further investigations. The problem is even less severe if the Fourier series (18) is finite as will be the case in our numerical example. The results (26), (28) enable us to proceed with the actual calculation of fidelity decay for ‘long’ and ‘short’ times, where the operators \bar{V} and Σ_t are dominant, respectively.

² Actually, in the off-diagonal matrix elements of the *Poisson bracket* we have oscillating functions of time $e^{(m'+m'') \cdot \omega t}$ which, for times longer than $\sim 1/\hbar$, can no longer reproduce the matrix element of the *commutator*.

3.1. Asymptotic regime of long times

Thus for sufficiently long times t , say longer than a certain $t_2 \propto \delta^{-1}$ such that the second term $\bar{V}t\tau\delta^2/2$ in BCH expansion (9), (17) dominates the first one $\Sigma_t\delta$, the fidelity (amplitude) can be written as

$$f(t) \cong \left\langle \exp\left(i\frac{\tau\delta^2}{2\hbar}\bar{V}t\right) \right\rangle, \quad \text{for } t > t_2. \quad (29)$$

Since both operators, Σ_t and \bar{V} , have well defined classical limits, it is clear that t_2 will generally *not depend* on \hbar ; however, it may depend on the precise structure of the initial state. Roughly it can be estimated semiclassically as

$$t_2 = \left(\frac{1}{|\bar{v}|} \sqrt{\sum_{m \neq 0} |\tilde{v}_m|^2} \right)_{\text{ef}} \frac{2}{\tau\delta} \quad (30)$$

where subscript ‘ef’ means *effective value* in the action space region of interest, i.e. where the initial state is distributed. Note that the actual timescale t_2 of the dominance of the second order can be in fact up to a factor $\sim \hbar^{-1/2}$ longer for coherent initial states and for sufficiently small perturbation, as explained in section 5.

The formula (29) can be transformed, following [6], into a very useful expression for the semiclassical analysis by, first, writing out the average as the trace in EBK basis $|\mathbf{n}\rangle$, second, using the fact that \bar{V} is diagonal in $|\mathbf{n}\rangle$ with eigenvalues $\cong \bar{v}(\hbar\mathbf{n})$, and, third, semiclassically approximating the sum $\sum_{\mathbf{n}}$ by an integral over the action space $\hbar^{-d} \int d^d\mathbf{j}$:

$$f(t) \cong \hbar^{-d} \int d^d\mathbf{j} \exp\left(i\frac{\tau\delta^2}{2\hbar}\bar{v}(\mathbf{j})t\right) D_\psi(\mathbf{j}), \quad t_2 < t < t^*, \quad (31)$$

$$D_\psi(\hbar\mathbf{n}) := |\langle \psi | \mathbf{n} \rangle|^2. \quad (32)$$

The last step is justified for (classically long) times up to t^* , such that the variation of the exponential in (31) across one Planck cell of diameter \hbar is small:

$$t^* = \frac{1}{|\partial_{\mathbf{j}}\bar{v}|_{\text{ef}}} \frac{1}{\tau\delta^2} \sim \hbar^0\delta^{-2}. \quad (33)$$

We note a strong formal similarity between the action space integral (ASI) representation of fidelity for a residual perturbing observable (31) and the ASI representation for a generic observable [6] for times up to $\sim \hbar^0\delta^{-1}$:

$$f_{\text{generic}}(t) \cong \hbar^{-d} \int d^d\mathbf{j} \exp\left(i\frac{\tau\delta}{\hbar}\bar{v}(\mathbf{j})t\right) D_\psi(\mathbf{j}). \quad (34)$$

This means that only $\bar{v}\delta$ has to be replaced by $\bar{v}\delta^2/2$ in the semiclassical analysis of formula (34) elaborated in [6]. This will be discussed in detail for the specific cases of coherent and random initial states in sections 5 and 6.

3.2. The plateau: linear response and beyond

For times t smaller than t_2 (30) the first term in the exponential of (17) dominates over the second one, so we may write the fidelity amplitude generally as

$$f(t) \cong \left\langle \exp\left(i\frac{\delta}{\hbar}\Sigma_t\right) \right\rangle, \quad \text{for } t < t_2. \quad (35)$$

Let us first discuss the regime of sufficiently small perturbation such that the fidelity is close to 1, i.e. the norm of the exponential is small, $\|\delta\Sigma_t/\hbar\| \ll 1$, so we can use the second order expansion of (35) which is precisely the linear response formula (8). We have to compute the uncertainty of the time integrated perturbation operator Σ_t . From the semiclassical expression for Σ_t (26) we can directly compute the expectation value

$$\langle \Sigma_t \rangle \cong \sum_n \sum_{m \neq 0} \tilde{v}_m(\hbar\mathbf{n})(1 - e^{im \cdot \omega(\hbar\mathbf{n})t}) \psi_{n-m} \psi_n^*, \quad (36)$$

where $\psi_n := \langle n | \psi \rangle$. Similarly, we compute the expectation value of its square

$$\langle \Sigma_t^2 \rangle \cong \sum_n \sum_{m, m' \neq 0} \tilde{v}_m(\hbar\mathbf{n}) \tilde{v}_{m'}(\hbar\mathbf{n}) (1 - e^{im \cdot \omega(\hbar\mathbf{n})t} - e^{im' \cdot \omega(\hbar\mathbf{n})t} + e^{i(m+m') \cdot \omega(\hbar\mathbf{n})t}) \psi_{n-m} \psi_{n+m'}^*. \quad (37)$$

For sufficiently many non-vanishing components ψ_n and/or for sufficiently large times $t > t_1$, and away from a certain resonance condition (all three conditions will be discussed in detail later) the terms with explicitly time dependent oscillating factors $\exp(im \cdot \omega(\hbar\mathbf{n})t)$ can be argued to give vanishing or (semiclassically) negligible contributions. In fact, the timescale t_1 will be determined by the condition that at typical later times, random phase approximation can be used in dealing with the exponentials in equations (36), (37). Thus the above expectation values should be time independent and equal to their time averages

$$\langle \Sigma_t \rangle \cong \overline{\langle \Sigma \rangle} = \sum_n \sum_{m \neq 0} \tilde{v}_m \psi_{n-m} \psi_n^*, \quad (38)$$

$$\langle \Sigma_t^2 \rangle \cong \overline{\langle \Sigma^2 \rangle} = \sum_n \left\{ \left| \sum_{m \neq 0} \tilde{v}_m(\hbar\mathbf{n}) \psi_{n-m} \right|^2 + \sum_{m \neq 0} |\tilde{v}_m(\hbar\mathbf{n}) \psi_{n-m}|^2 \right\}. \quad (39)$$

This gives us a prediction that, after following a classical decay up to the short time t_1 , the fidelity should reach a constant value—a plateau—and stay there up to time t_2 . The linear response value of the fidelity at the plateau is

$$F(t) \cong 1 - \frac{\delta^2}{\hbar^2} \{ \overline{\langle \Sigma^2 \rangle} - \overline{\langle \Sigma \rangle}^2 \} + \mathcal{O}(\delta^4), \quad \text{for } t_1 < t < t_2. \quad (40)$$

Further, we can easily go beyond the linear response approximation by expanding the formula (35) to all orders in δ . For this, we have to calculate the powers of the operator Σ_t :

$$\Sigma_t^k \cong \sum_{m_1, \dots, m_k \neq 0} \prod_{l=1}^k \tilde{v}_{m_l}(\mathbf{J}) e^{im_l \cdot \Theta} (1 - e^{im_l \cdot \omega(\mathbf{J})t}). \quad (41)$$

We shall use two facts in order to carry through the calculation:

- (i) For leading semiclassical order the operator ordering is not important.
- (ii) Oscillatory time dependent terms, which in any case average to zero, typically give exponentially damped or semiclassically small overall contributions when used in expectation values.

Thus we shall approximate Σ_t^k by its time average $\overline{\Sigma^k}$, and the latter is calculated by selecting from the product (41) only the combinations of multi-indices which sum up to zero, $\mathbf{m}_1 + \dots + \mathbf{m}_l = \mathbf{0}$. Let us write the time average of the operator Σ_t in terms of an explicit function of canonical operators:

$$\begin{aligned} \overline{\Sigma} &\cong \tilde{v}(\mathbf{J}, \Theta), \\ \tilde{v}(\mathbf{j}, \theta) &:= \sum_{\mathbf{m} \neq \mathbf{0}} \tilde{v}_{\mathbf{m}}(\mathbf{j}) e^{i\mathbf{m} \cdot \theta}. \end{aligned} \quad (42)$$

Then we calculate

$$\begin{aligned} \overline{\Sigma^k} &\cong \overline{\Sigma}^k + \sum_{l=2}^k (-1)^l \binom{k}{l} \overline{\Sigma}^{k-l} \sum_{\mathbf{m}_1, \dots, \mathbf{m}_l \neq \mathbf{0}} \delta_{\mathbf{m}_1 + \dots + \mathbf{m}_l} \tilde{v}_{\mathbf{m}_1}(\mathbf{J}) \cdots \tilde{v}_{\mathbf{m}_l}(\mathbf{J}) \\ &= \frac{1}{(2\pi)^d} \sum_{l=0}^k (-1)^l \binom{k}{l} \tilde{v}(\mathbf{J}, \Theta)^{k-l} \int d^d \mathbf{x} [\tilde{v}(\mathbf{J}, \mathbf{x})]^l. \end{aligned} \quad (43)$$

Please observe that Θ is an angle operator (which always stands in the exponential, so it is well defined), and \mathbf{x} is a d -dimensional integration variable, and that in order to write (43) we have used an integral representation of the Kronecker symbol: $\delta_{\mathbf{m}} = (2\pi)^{-d} \int d^d \mathbf{x} e^{i\mathbf{m} \cdot \mathbf{x}}$. Now it is straightforward to compute the power series $\sum_{k=0}^{\infty} (i\delta/\hbar)^k \overline{\Sigma^k} / k!$ by changing the summation variables to $k-l$ and k , yielding a product of two exponentials:

$$\overline{\exp\left(\frac{i\delta}{\hbar} \Sigma\right)} \cong \exp\left(\frac{i\delta}{\hbar} \tilde{v}(\mathbf{J}, \Theta)\right) \int \frac{d^d \mathbf{x}}{(2\pi)^d} \exp\left(-\frac{i\delta}{\hbar} \tilde{v}(\mathbf{J}, \mathbf{x})\right), \quad (44)$$

and the plateau of fidelity (amplitude) is the expectation value of this operator

$$f_{\text{plateau}} \cong \int \frac{d^d \mathbf{x}}{(2\pi)^d} \left\langle \exp\left(\frac{i\delta}{\hbar} \tilde{v}(\mathbf{J}, \Theta)\right) \exp\left(-\frac{i\delta}{\hbar} \tilde{v}(\mathbf{J}, \mathbf{x})\right) \right\rangle; \quad (45)$$

$f(t) = f_{\text{plateau}}$ for $t_1 < t < t_2$. The very existence of such a plateau of fidelity (high fidelity for small δ/\hbar) is very interesting and distinct property of quantum dynamics. Note that the timescale t_1 only depends on the unperturbed dynamics, namely on the property of the operator Σ_t , so it cannot depend on the strength of the perturbation, $t_1 = \mathcal{O}(\delta^0)$. Thus the range of the plateau, i.e. $t_2/t_1 \propto \delta^{-1}$, can become arbitrarily large for small δ . The formula (45) becomes very useful whenever one is able to semiclassically compute the quantum expectation value in terms of classical phase space integrals. We shall present this derivation for two extremal cases of *coherent* and *random* initial states, in sections 5 and 6, respectively.

Note that the formulae (44), (45) may be very useful in a general case whenever one has to calculate an expectation value of the form $\langle \exp(i\frac{\delta}{\hbar} \Sigma_t) \rangle$ where Σ_t is a time integrated quasi-periodic process with a zero time average.

4. Numerical example: integrable top

For numerical illustration of the above theory we take a spin system with the following one-time-step unitary propagator

$$U_0 = \exp\left\{-iS\frac{\alpha}{2}\left(\frac{S_z}{S} - \beta\right)^2\right\}, \quad (46)$$

with parameters α and β . S_k , $k = x, y, z$, are standard quantum angular momentum operators with a fixed magnitude S of angular momentum and with the $SU(2)$ commutator $[S_k, S_l] = i\epsilon_{klm}S_m$.

The semiclassical limit is obtained by letting $S \rightarrow \infty$ while the classical angular momentum $\hbar S = 1$ is kept fixed, so the effective Planck constant is given by $\hbar = 1/S$. The classical map corresponding to the one-time-step propagator U_0 can be obtained from the Heisenberg equations of angular momentum operators in the $S \rightarrow \infty$ limit. Defining by $(x, y, z) = (S_x, S_y, S_z)/S$ a point on a unit sphere, we obtain a classical area preserving map:

$$\begin{aligned} x_{t+1} &= x_t \cos(\alpha(z_t - \beta)) - y_t \sin(\alpha(z_t - \beta)), \\ y_{t+1} &= y_t \cos(\alpha(z_t - \beta)) + x_t \sin(\alpha(z_t - \beta)), \\ z_{t+1} &= z_t. \end{aligned} \quad (47)$$

This classical map represents a twist around the z -axis. We note that it corresponds to the stroboscopic map (19) with an arbitrary unit of time, so we put $\tau = 1$, for an integrable system with the Hamiltonian $h_0(j) = \frac{1}{2}\alpha(j - \beta)^2$ generating a frequency field

$$\omega(j) = \frac{dh_0(j)}{dj} = \alpha(j - \beta). \quad (48)$$

Here we used a canonical transformation from a unit sphere to an action–angle pair $(j, \theta) \in [-1, 1] \times [0, 2\pi)$, namely

$$x = \sqrt{1 - j^2} \cos \theta, \quad y = \sqrt{1 - j^2} \sin \theta, \quad z = j. \quad (49)$$

Now we perturb the Hamiltonian by periodic kicking with a transverse pulsed magnetic field in the x direction,

$$h_\delta(j, \theta, \tau) = \frac{1}{2}\alpha(j - \beta)^2 + \delta\sqrt{1 - j^2} \cos \theta \sum_{k=-\infty}^{\infty} \delta(\tau - k). \quad (50)$$

The perturbed quantum evolution is given by a product of two unitary propagators

$$U_\delta = U_0 \exp(-i\delta S_x), \quad (51)$$

so the perturbation generator is

$$V = S_x/S. \quad (52)$$

The classical perturbation has only one Fourier component, namely

$$v(j, \theta) = \sqrt{1 - j^2} \cos \theta, \quad v_{\pm 1}(j) = \frac{1}{2}\sqrt{1 - j^2}, \quad (53)$$

whereas $v_0 \equiv 0$ indicating that the time average vanishes: $\bar{v} = 0$, and $\bar{V} = 0$.

In our numerical illustrations two different types of initial state will be used. $SU(2)$ coherent wavepackets are used to probe the correspondence with the classical fidelity, while random states are used to investigate the other end-states without a classical correspondence. The parameter α in U_0 (46) will always be set to $\alpha = 1.1$, while $\beta = 0$ for coherent initial states, and $\beta = 0$ and $\beta = 1.4$ for random initial states. The reason for choosing non-zero shift β for random states will be explained later. We should stress that we have performed calculations also for other choices of regular U_0 , also in the KAM regime, e.g. for precisely the same model and parameter values as used in [10], and obtained qualitatively the same results as for the presented case of unperturbed dynamics. The coherent state written in the canonical eigenbasis $|m\rangle$ of the operator S_z and centred at the position $\mathbf{n} = (\sin \vartheta^* \cos \varphi^*, \sin \vartheta^* \sin \varphi^*, \cos \vartheta^*)$ on a unit sphere is

$$|\vartheta^*, \varphi^*\rangle = \sum_{m=-S}^S \binom{2S}{S+m}^{1/2} \cos(\vartheta^*/2)^{S+m} \sin(\vartheta^*/2)^{S-m} e^{-im\varphi^*} |m\rangle. \quad (54)$$

The corresponding classical density reads [11]

$$\rho_{\text{cl}}(\vartheta, \varphi) = \frac{4S+1}{4\pi} \exp\{-S((\vartheta - \vartheta^*)^2 + (\varphi - \varphi^*)^2 \sin^2 \vartheta)\}. \quad (55)$$

In the numerical experiments reported below the coherent initial state will always be positioned at the point $(\vartheta^*, \varphi^*) = (1, 1)$.

5. Semiclassical asymptotics: coherent initial state

Let us now study an important specific case of a (generally squeezed) coherent initial state which can be written in the EBK basis as a general Gaussian centred around a phase space point $(\mathbf{j}^*, \boldsymbol{\theta}^*)$:

$$\langle \mathbf{n} | \mathbf{j}^*, \boldsymbol{\theta}^* \rangle \cong \left(\frac{\hbar}{\pi}\right)^{d/4} |\det \Lambda|^{1/4} \exp\left\{-\frac{1}{2\hbar}(\hbar \mathbf{n} - \mathbf{j}^*) \cdot \Lambda (\hbar \mathbf{n} - \mathbf{j}^*) - i \mathbf{n} \cdot \boldsymbol{\theta}^*\right\}, \quad (56)$$

with Λ being a positive symmetric $d \times d$ matrix of squeezing parameters. Note that the shape of the coherent state is generally only asymptotically Gaussian, as $\hbar \rightarrow 0$, due to the cyclic and discrete nature of the coordinates $\boldsymbol{\theta}$ and \mathbf{j} , respectively. Let us also write out the *structure function* (32) of our coherent state (56):

$$D_\psi(\mathbf{j}) \cong \left(\frac{\hbar}{\pi}\right)^{d/2} |\det \Lambda|^{1/2} \exp\left\{-\frac{1}{\hbar}(\mathbf{j} - \mathbf{j}^*) \cdot \Lambda (\mathbf{j} - \mathbf{j}^*)\right\} \quad (57)$$

which is normalized as $\hbar^{-d} \int d^d \mathbf{j} D_\psi(\mathbf{j}) = 1$.

For example, for an $SU(2)$ coherent state of a quantum top (54), written in the asymptotic form (56), the squeezing parameter reads $\Lambda = 1/\sin^2 \vartheta^*$.

5.1. The plateau: linear response and beyond

In the regime of linear response, valid for sufficiently small δ , we simply evaluate the general expressions (36), (37) for the particular case of a coherent initial state (56); that is, we write $\psi_n = \langle \mathbf{n} | \mathbf{j}^*, \boldsymbol{\theta}^* \rangle$. First we will show that the time dependent terms in expectation values of the powers of Σ_t do indeed vanish for $t > t_1$ as stated in section 3.2. We recall the assumption that the perturbation $v(\mathbf{j}, \boldsymbol{\theta})$ is sufficiently smooth, e.g. analytic in $\boldsymbol{\theta}$, so that the Fourier coefficients

$v_m(\mathbf{j})$ decrease sufficiently fast, e.g. exponentially, or only a finite number of $v_m(\mathbf{j})$ is non-vanishing. What we actually need here is that an effective number of Fourier components are smaller than the width of a wavepacket which is $\sim \hbar^{-1/2}$. This means that within the range of Fourier series over \mathbf{m} , or \mathbf{m}' , we can make the approximation

$$\psi_{n-m}^* \psi_{n+m'} \cong D_\psi(\hbar \mathbf{n}) e^{-i(m+m') \cdot \boldsymbol{\theta}^*}. \quad (58)$$

Let us estimate the general time dependent term of expressions (36), (37), where all factors with a non-singular classical limit are combined together and denoted as $g(\mathbf{j})$, by means of expanding the frequency around the centre of the packet $\omega(\mathbf{j}^* + \mathbf{x}) = \omega(\mathbf{j}^*) + \Omega \mathbf{x} + \dots$, where Ω is a matrix $\Omega_{kl} = \partial \omega_k(\mathbf{j}^*) / \partial j_l$, followed by d -dimensional Gaussian integration:

$$\begin{aligned} \sum_n g(\hbar \mathbf{n}) e^{im \cdot \omega(\hbar \mathbf{n})t} D_\psi(\hbar \mathbf{n}) &\cong \hbar^{-d} \int d^d \mathbf{j} g(\mathbf{j}) e^{im \cdot \omega(\mathbf{j})t} D_\psi(\mathbf{j}) \\ &\cong g(\mathbf{j}^*) e^{im \cdot \omega(\mathbf{j}^*)t} \left(\frac{\hbar}{\pi}\right)^{d/2} |\det \Lambda|^{1/2} \int d^d \mathbf{x} \exp\left(-\frac{1}{\hbar} \mathbf{x} \cdot \Lambda \mathbf{x} + it \mathbf{m} \cdot \Omega \mathbf{x}\right) \\ &= g(\mathbf{j}^*) e^{im \cdot \omega(\mathbf{j}^*)t} \exp\left(-\frac{\hbar t^2}{4} \mathbf{m} \cdot \Omega \Lambda^{-1} \Omega^T \mathbf{m}\right). \end{aligned} \quad (59)$$

We see that all these terms decay to zero with Gaussian envelopes with the longest timescale estimated as

$$t_1 = \left(\frac{\hbar}{4} \min_{\mathbf{m} \neq \mathbf{0}} (\mathbf{m} \cdot \Omega \Lambda^{-1} \Omega^T \mathbf{m})\right)^{-1/2} \propto \hbar^{-1/2}. \quad (60)$$

Note that the decay of (59) is absent if $\Omega = 0$, e.g. in the case of a d -dimensional harmonic oscillator. There may also be a general problem with the formal existence of the scale t_1 (60) if the derivative matrix Ω is singular, but this may not actually affect the fidelity for sufficiently quickly converging or finite Fourier series (18).

Thus we have shown that for $t > t_1$, expectation values (36), (37) are indeed given by time independent expressions (38), (39), which in the case of a coherent initial state (56) evaluate to

$$\langle \Sigma_t \rangle \cong \sum_{\mathbf{m} \neq \mathbf{0}} \tilde{v}_m(\mathbf{j}^*) e^{im \cdot \boldsymbol{\theta}^*} = \tilde{v}(\mathbf{j}^*, \boldsymbol{\theta}^*), \quad (61)$$

$$\langle \Sigma_t^2 \rangle \cong (\tilde{v}(\mathbf{j}^*, \boldsymbol{\theta}^*))^2 + \sum_{\mathbf{m} \neq \mathbf{0}} |\tilde{v}_m(\mathbf{j}^*)|^2. \quad (62)$$

The variance $\nu = \langle \Sigma_t^2 \rangle - \langle \Sigma_t \rangle^2$ which determines the plateau in the fidelity (8) is the second term on the RHS of equation (62). In terms of the original Fourier coefficients (27), the final linear response result reads

$$\begin{aligned} F_{\text{coh}}(t) &= 1 - \frac{\delta^2}{\hbar^2} \nu_{\text{coh}} + \mathcal{O}(\delta^4), \quad t > t_1, \\ \nu_{\text{coh}} &= \sum_{\mathbf{m} \neq \mathbf{0}} \frac{\tau^2 |v_m(\mathbf{j}^*)|^2}{4 \sin^2(\frac{1}{2} \mathbf{m} \cdot \boldsymbol{\omega}(\mathbf{j}^*))}. \end{aligned} \quad (63)$$

Beyond the linear response approximation, the value of the plateau can be computed by applying a general formula for f_{plateau} (45). We shall make use of the fact that for coherent states we have the expectation value

$$\langle \exp(-i\delta/\hbar)g(\mathbf{J}, \boldsymbol{\Theta}) \rangle \cong \exp(-i\delta/\hbar)g(\mathbf{j}^*, \boldsymbol{\theta}^*)$$

for some smooth function g , provided that the diameter of the wavepacket, $\sim\sqrt{\hbar}$, is smaller than the oscillation scale of the exponential, $\sim\hbar/\delta$, i.e. provided that $\delta \ll \hbar^{1/2}$. Then the squared modulus of f_{plateau} (45) is rewritten as

$$F_{\text{plateau}} \cong \frac{1}{(2\pi)^{2d}} \left| \int d^d \mathbf{x} \exp\left(-\frac{i\delta}{\hbar} \tilde{v}(\mathbf{j}^*, \mathbf{x})\right) \right|^2. \quad (64)$$

The expression for ν_{coh} , namely (63), is of course just the lowest order expansion of F_{plateau} (64). It is interesting to note that the angle $\boldsymbol{\theta}^*$ does not affect the probability F_{plateau} as it only rotates the phase of the amplitude f_{plateau} .

For smaller times $t < t_1 \propto \hbar^{-1/2}$ the quantum fidelity is expected to follow the classical fidelity as defined by the overlap of two initially Gaussian classical phase space densities evolved under slightly different quasi-regular time evolutions (see [6] for a definition and linear response treatment of the classical fidelity). More precisely, the quantum fidelity can be written in two equivalent ways as

$$F(t) = (2\pi\hbar)^d \int d^d \mathbf{q} d^d \mathbf{p} W_{U_0^t|\psi\rangle}(\mathbf{q}, \mathbf{p}) W_{U_\delta^t|\psi\rangle}(\mathbf{q}, \mathbf{p}) \quad (65)$$

$$= (2\pi\hbar)^d \int d^d \mathbf{q} d^d \mathbf{p} W_{|\psi\rangle}(\mathbf{q}, \mathbf{p}) W_{M_\delta(t)|\psi\rangle}(\mathbf{q}, \mathbf{p}) \quad (66)$$

where $W_{|\psi\rangle}(\mathbf{q}, \mathbf{p})$ is the *Wigner function* of some state $|\psi\rangle$. The corresponding classical fidelity is defined by the same formula if $(2\pi\hbar)^{d/2} W_{U^t|\psi\rangle}(\mathbf{q}, \mathbf{p})$ is replaced by the evolving classical phase space density, a solution of the corresponding classical Liouville equation, with the initial condition $(2\pi\hbar)^{d/2} W_{|\psi\rangle}(\mathbf{q}, \mathbf{p})$, which is proportional to the Wigner function of the initial state $|\psi\rangle$. Of course, this only makes sense if the function $W_{|\psi\rangle}(\mathbf{q}, \mathbf{p})$ is strictly non-negative with the result that it corresponds to some classical state, such as for a coherent state where it is a Gaussian. Indeed, time $t_1 \sim \hbar^{-1/2}$ may also be interpreted as the *integrable Ehrenfest time* up to which phase space point-like quantum–classical correspondence will hold. That is, it is consistent with the time needed for a minimal uncertainty wavepacket of diameter $\sim\hbar^{1/2}$ to spread ballistically over a region of the classical size ($\sim\hbar^0$) of an invariant torus. After this time, the quantum wavepacket will start to coherently interfere with itself, e.g. its Wigner function will develop negative values, so the strict quantum–classical correspondence will cease (see section 5.4). Therefore we expect initial agreement between the classical and the quantum fidelity up to time t_1 and after that the classical fidelity of a regular dynamics with a residual perturbation decays with a power law $\propto(\delta\hbar^{-1/2}t)^{-d}$ (factor $\hbar^{1/2}$ comes from the size of the corresponding classical density; see [8, 9]) whereas the quantum fidelity *freezes to a constant value* as computed by our semiclassical theory (64).

This picture is nicely confirmed by the numerical experiment with a quantized integrable top (46) as shown in figure 1, where we choose zero shift: $\beta = 0$. Agreement between the classical and quantum fidelity up to $t_1 \sim \sqrt{S}$ ($\hbar = 1/S$) can be nicely observed. After t_1 the fidelity stays constant up to t_2 , the point where fidelity again starts to decrease. This second time-scale t_2 will be discussed in the next subsection. The value of the plateau can be

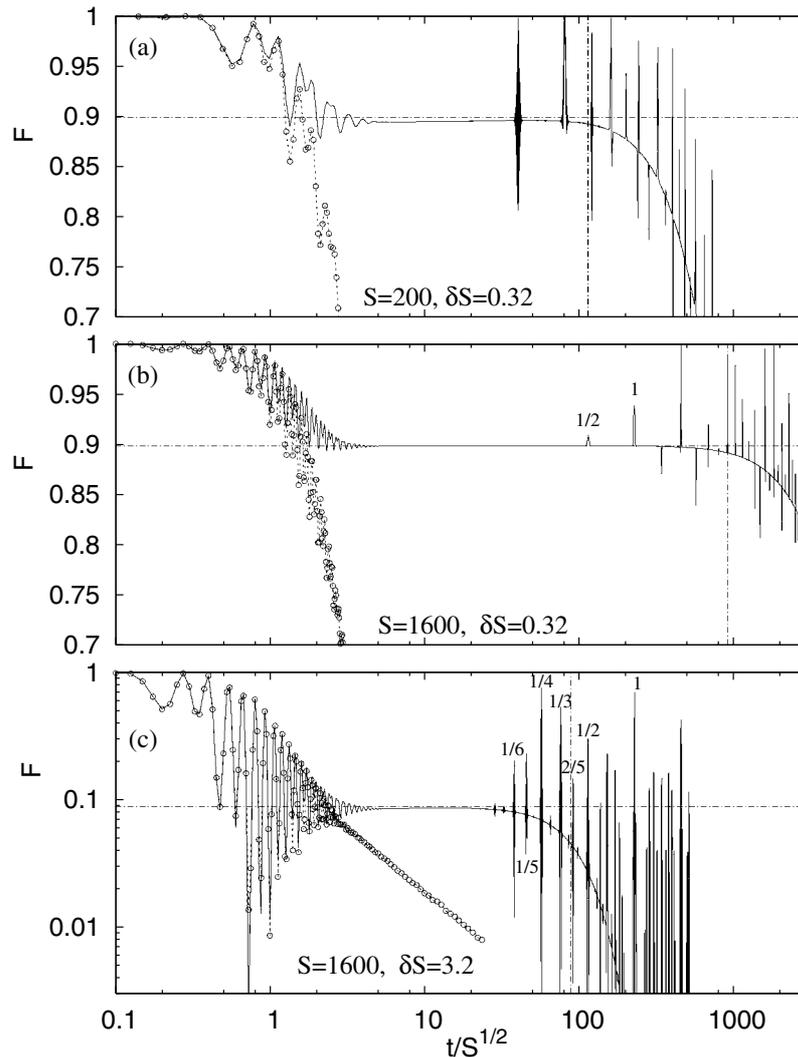


Figure 1. The short time decay of the fidelity for a quantized top $\alpha = 1.1$, $\beta = 0$ is shown for the coherent initial state, for $S = 200$ (a) and $S = 1600$ (b), with a fixed product $\delta S = 0.32$; it is well described by the linear response. In (c) we show $S = 1600$ and stronger perturbation with $\delta S = 3.2$. Note that the time axis is rescaled as t/t_1 . Symbols connected with dashed curves denote the corresponding classical fidelity. The horizontal chain line denotes the theoretical value of the plateau (67), while the vertical chain line denotes the estimated theoretical value for t_2 , given by (71). In (b), (c) we also indicate fractional $2\pi k/p$ resonances with k/p marked in the figure (see the text for details).

calculated specifically for our model by means of the semiclassical expression for F_{plateau} (64) and using Fourier modes of our numerical model (53). We get $\tilde{v}(j, \theta) = -\frac{1}{2}\sqrt{1-j^2} \sin(\theta - \frac{1}{2}\alpha j)/\sin(\alpha j/2)$ and the integral occurring in F_{plateau} is elementary and gives

$$F_{\text{plateau}} = J_0^2 \left(\delta S \frac{\sqrt{1-j^{*2}}}{2 \sin(\alpha j^*/2)} \right), \quad (67)$$

with J_0 being the zero order Bessel function. Agreement with this theory is excellent both in the linear response regime (figures 1(a), (b)) and also for strong perturbation (figure 1(c)). Observe also a power law decay of the classical fidelity $F_{cl} \sim (t\delta/\hbar)^{-1}$ beyond the regular Ehrenfest time $t > t_1$ for strong perturbation in figure 1(c).

Actually the calculation of F_{plateau} can be generalized to any perturbation with a single non-zero Fourier mode $\pm m_0$ with the result

$$F_{\text{plateau}} = J_0^2 \left(\frac{\tau \delta}{\hbar} \frac{|v_{m_0}(j^*)|}{\sin\{m_0 \cdot \omega(j^*)/2\}} \right), \quad (68)$$

whereas for a more general multi-mode perturbations we have to evaluate the integral (64) numerically.

5.2. Asymptotic regime of long times

After a sufficiently long time t_2 of order δ^{-1} the second order term in BCH expansion (9) will start to dominate over the echo operator, so the fidelity can be computed with our semiclassical formula (31). The straightforward calculation follows exactly the one for a generic perturbation in [6], paragraph 2.2.2, where $\bar{v}\delta$ has to be replaced by $\bar{v}\delta^2/2$, so we shall not repeat it here. Such a Gaussian approximation is justified provided that the stationary point of the exponent is not moved appreciably from the centre j^* of the packet (57). This implies that $t\delta^2 \ll 1$ which is the same condition as required by replacing the sum over quantum numbers by the integral over the action space (31). The final result reads

$$f_{\text{coh}}(t) \cong \exp \left(-(\mathbf{u} \cdot \Lambda^{-1} \mathbf{u}) \frac{\delta^4 t^2}{16\hbar} + \frac{i\bar{v}(j^*)\delta^2 \tau t}{2\hbar} \right), \quad \mathbf{u} := \tau \frac{\partial \bar{v}(j^*)}{\partial \mathbf{j}}, \quad (69)$$

where the vector \mathbf{u} is just a gradient of the classical observable \bar{v} at the centre of the wavepacket. Thus we have derived a Gaussian decay of the fidelity

$$F_{\text{coh}}(t) = \exp \left\{ - \left(\frac{t}{t_{\text{coh}}} \right)^2 \right\}, \quad t_{\text{coh}} = (\mathbf{u} \cdot \Lambda^{-1} \mathbf{u})^{-1/2} \frac{(8\hbar)^{1/2}}{\delta^2}, \quad (70)$$

on a timescale $t_{\text{coh}} \propto \hbar^{1/2} \delta^{-2}$. Indeed, it can now be checked that in the semiclassical regime of small \hbar the fidelity decays well before the time limit (33), $t^* \sim \hbar^0 \delta^{-2}$, of our approximations. Formula (70) can only be expected to be accurate provided that the plateau is close to 1 and hence described within the linear response approximation. Only in such a case can the effect of the first term Σ_t in the exponential of the echo operator (17) really be neglected; in the opposite case we can correct the Gaussian (70) by multiplying it with the plateau value and adjusting the coefficient in the exponential (see e.g. figure 2(c)).

In such a *regime of small perturbation*, $\delta < v_{\text{coh}}^{-1/2} \hbar$, we determine the crossover time t_2 by comparing the linear response formula (63) with the decay law (70), namely $1 - (t_2/t_{\text{coh}})^2 = 1 - \delta^2 v_{\text{coh}}/\hbar^2$. For stronger perturbation, namely up to $\delta \sim \sqrt{\hbar}$, timescale t_2 can be simply estimated by t_{coh} , so we have a uniform estimate

$$t_2 = \min \left\{ 1, \frac{\delta}{\hbar} v_{\text{coh}}^{1/2} \right\} t_{\text{coh}} = \min \{ \text{constant} \times \hbar^{-1/2} \delta^{-1}, \text{constant} \times \hbar^{1/2} \delta^{-2} \}. \quad (71)$$

We note that the crossover time t_2 , for coherent initial states and for a small perturbation $\delta < v_{\text{coh}}^{-1/2} \hbar$, is in fact longer by a factor $\hbar^{-1/2}$ than the estimate (30). This is due to the fact

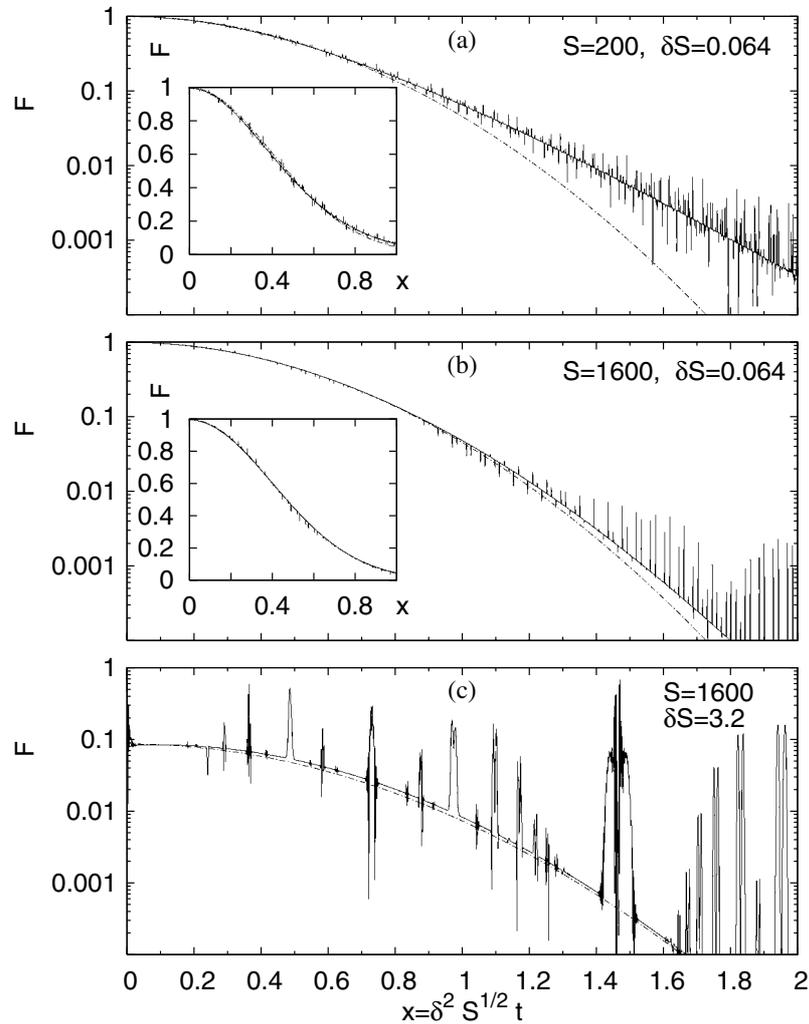


Figure 2. Long time ballistic decay of the fidelity for a quantized top with $\alpha = 1.1$, $\beta = 0$ and a coherent initial state is shown for cases $S = 200$ (a) and $S = 1600$ (b), of weak perturbation $\delta S = 0.64$, and for strong perturbation: $S = 1600$, $\delta S = 3.2$ (c). Chain curves indicate the theoretical Gaussian (70) with analytically computed coefficients, except in case (c), where we multiply the theoretical Gaussian decay by a prefactor 0.088 which is equal to the theoretical value of the plateau (67), and rescale the exponent of the Gaussian by a factor 0.8 taking into account the effect of the non-small first term in the exponent of (17). Note that in the limit $S \rightarrow \infty$ the agreement with the semiclassical theory improves and that the size of the resonant spikes is of the same order as the drop in the linear response plateau. The insets show the data and the theory on the normal scale.

that coherent states are strongly localized in action coordinates (quantum numbers) for small \hbar . Therefore, for small δ , the operator $\bar{V}t\delta^2$, although it may already be dominating $\Sigma_t\delta$ in norm, will only effectively rotate the overall phase of a coherent initial state since it is diagonal in $|n\rangle$ and thus will not (yet) affect the fidelity. So the estimate (30) is expected to be valid only for initial states whose relative support in the quantum number lattice is not shrinking as $\hbar \rightarrow 0$. It is interesting that for the strongest allowed perturbation $\delta_{\max} = \text{constant} \times \hbar^{1/2}$ for our semiclassical theory to be valid, the estimate (71) agrees with the general estimate (30), $t_2 \sim 1/\delta$.

Timescale t_2 can be seen in figure 1 as the point of departure of fidelity from the plateau value. Using our model and the position of the initial coherent state this can be calculated to be $t_2 = \min\{0.57\sqrt{S}/\delta, 0.57/(\delta^2\sqrt{S})\}$ (the similarity of the numerical prefactors is just a coincidence) which is shown with vertical chain lines in figure 1. The theoretical position of t_2 is shown with a vertical chain line and is given by t_{coh} for a strong perturbation $\delta S = 3.2$ in figure 1(c) while it is $t_{\text{coh}}v_{\text{coh}}^{1/2}\delta/\hbar$ in figures 1(a), (b). The long time decay of the fidelity is shown in figure 2. The theoretical Gaussian decay (70), shown with a chain curve, is again confirmed with an analytical formula for the decay time $t_{\text{coh}} = 0.57\delta^{-2}\hbar^{1/2}$ evaluated at the particular position of the packet. Note that we do not have any fitting parameters, except in the case of a strong perturbation ($\delta S \gg 1$; figure 2(c)) where the prefactor and the exponent of a Gaussian had to be slightly adjusted due to the non-negligible effect of the first term in (17) (see the caption for details). Quite prominent in figures 1 and 2 are also the ‘spikes’ occurring at regular intervals, where the fidelity suddenly increases or wildly oscillates. These will be called the *echo resonances* and are particular to one-dimensional systems.

We should remark that, although we obtain asymptotically Gaussian decay of fidelity for a single coherent initial state, one may be interested in an *effective fidelity* averaged with respect to phase space positions of the initial coherent state [10]. In such a case one may typically get a power law decay due to possible points in the phase space where the theoretical expression for t_{coh} diverges (at the positions of zeros of $u(j^*)$ of equation (69)), but still on a timescale $\propto \delta^{-2}$. Note that this *effective power law decay* is a general scenario and is not particular to the case of vanishing time-average perturbation (in the case of $\bar{V} \neq 0$ the decay time scales as δ^{-1}).

5.3. Echo resonances in one dimension

Let us now discuss the behaviour of the fidelity for initial wavepackets in the regime of the linear response approximation in some more detail. We shall consider possible deviations from the random phase approximation in the time dependent exponentials of equations (36), (37) which have been invoked previously in order to derive the time independent terms (38), (39) of the fidelity plateau (40), and also (45). Specifically we will explain the resonances observed e.g. in figure 1.

For such a resonance to occur the phases of (36), (37) have to build up in a constructive way and this is clearly impossible in a generic case, unless:

- (i) We have one dimension $d = 1$, so we sum up over a one-dimensional array of integers n in the action space³.
- (ii) The wavepacket is localized over a classically small region of the action space/lattice such that a variation of the frequency derivative $d\omega(j)/dj$ over this region is sufficiently small.

³ In more than one dimension we would clearly need a strong condition on commensurability of frequency derivatives over the entire region of the action lattice where the initial state is supported.

The quantitative conditions for the occurrence together with the strength and the shape of such resonances are discussed below.

In this subsection we thus consider a one-dimensional case, $d = 1$. Again we study time dependent terms of (36), (37) which can all be cast into a general form (59); however, now the time is not small enough to enable the sum over the quantum number to be estimated by an integral. In contrast, we seek a condition such that the consecutive phases in the exponential build up an interference pattern.

5.3.1. 2π resonance. Let us expand the frequency around the centre of the packet

$$\omega(j) = \omega^* + (j - j^*)\omega' + \frac{1}{2}(j - j^*)^2\omega'' + \dots \quad (72)$$

where

$$\omega^* = \omega(j^*), \quad \omega' = \frac{d\omega(j^*)}{dj}, \quad \omega'' = \frac{d^2\omega(j^*)}{dj^2}. \quad (73)$$

The phases in the sums of the form (59) come into resonance, for $m = 1$ and hence for any higher $m \geq 1$, when they change by 2π per quantum number, which happens at time t_r :

$$\hbar\omega't_r = 2\pi, \quad t_r = \frac{2\pi}{\hbar\omega'} \quad (74)$$

and its integer multiples⁴. Let time t be close to kt_r , $k \in \mathbb{Z}$, and write $t = kt_r + t'$ where $t' \ll t_r$, so

$$\hbar\omega't' \ll 2\pi. \quad (75)$$

Now we can estimate the general time dependent term (59), where $g(j)$ is again a smooth classical \hbar independent function of j representing a suitable combination of Fourier coefficients $v_m(j)$, by: (i) shifting the time variable to t' , (ii) incorporating the resonance condition (74), and (iii) approximating the resulting sum by an integral, due to the smallness of t' , given by (75), which is a simple Gaussian:

$$\begin{aligned} \sum_n g(\hbar n) e^{im\omega(\hbar n)t} D_\psi(\hbar n) &\cong \sum_n g(\hbar n) e^{im(\omega^* + (\hbar n - j^*)\omega' + \frac{1}{2}(\hbar n - j^*)^2\omega'')(kt_r + t')} D_\psi(\hbar n) \\ &= e^{im\omega^*t} \sum_n g(\hbar n) e^{im((\hbar n - j^*)\omega't' + \frac{1}{2}(\hbar n - j^*)^2\omega''t)} D_\psi(\hbar n) \\ &\cong e^{im\omega^*t} g(j^*) \sqrt{\frac{\Lambda}{\pi\hbar}} \int dj e^{im((j - j^*)\omega't' + \frac{1}{2}(j - j^*)^2\omega''t) - (\Lambda/\hbar)(j - j^*)^2} \\ &= e^{im\omega^*t} g(j^*) \left(1 - i\frac{\hbar m\omega''t}{2\Lambda}\right)^{-1/2} \exp\left(-\frac{\hbar m^2\omega'^2t'^2}{4\Lambda} \frac{1 + i\hbar m\omega''t/(2\Lambda)}{1 + (\hbar m\omega''t/(2\Lambda))^2}\right). \end{aligned} \quad (76)$$

From this calculation we deduce the quantitative condition for the appearance and the shape of the resonance. Let $\Delta_j = \langle (J - j^*)^2 \rangle^{1/2} = \sqrt{\hbar/(2\Lambda)}$ denote the action width of the wavepacket. Physically, we need that the coherence of linearly increasing phases is not lost along the size of the wavepacket, i.e.

⁴ It is interesting to note that these resonant times correspond precisely to the condition for *revivals* of the wavepacket in the *forward* evolution (apart from a phase space translation) studied in [14].

$$\zeta := m\omega''t\Delta_j^2 = \frac{\hbar m\omega''t}{2\Lambda} < 2\pi. \quad (77)$$

ζ is precisely the coefficient appearing in the square-root prefactor and the exponential of the resonance profile (76). Indeed we see that with increasing ζ the squared modulus of the peak of the resonance (at $t' = 0$) dies out as $(1 + \zeta^2)^{-1/2}$. We note that ζ increases with increasing order k of the resonance, since $t \approx kt_r$, so

$$\zeta = km \frac{\pi \omega''}{\Lambda \omega'}. \quad (78)$$

Therefore we may get strong and numerous resonances, i.e. small ζ , provided that either the second derivative ω'' is small, or the initial state is squeezed such that $\Lambda \gg 1$. For example, if the second derivative vanishes everywhere, $\omega'' \equiv 0$, then the resonances may appear even for extended states. This is the case for our numerical model, where resonances can be seen also for a random state in figure 6.

From (76) we read that the temporal profile in such a fidelity resonance has the shape of a Gaussian of effective width

$$\Delta_t = \frac{\sqrt{1 + \zeta^2}}{m\omega' \Delta_j}, \quad (79)$$

modulated with an oscillation of frequency $\cong \omega^*$. Hence in order for the effect of the fidelity resonance to be felt, time t has to be within Δ_t of the centre of the resonance kt_r . In the semiclassical limit, the resonance positions scale as $t_r \propto \hbar^{-1}$, while their widths grow only as $\Delta_t \propto \hbar^{-1/2}$, so they are well separated. Also, with increasing order k the magnitudes of the peaks of the resonances decrease as $\sim k^{-1/2}$, while their widths increase as $\sim k$, so they will eventually, at $k \propto \hbar^{-1/2}$, start to overlap. This will happen at time $\sim \hbar^{-3/2}$ which is smaller than $\sim t_2$ provided that $\delta < \nu^{-1/2}\hbar$.

The resonance described in this paragraph, which will be called a 2π resonance, affects *all* time dependent terms of (36), (37), but its precise shape depends on coefficients $\tilde{v}_m(j^*)e^{im\theta^*}$. However, it is important to note that the fidelity can be explicitly calculated close to the centre of the resonance, $t' \ll \Delta_t$, where the Gaussian factor of the rightmost expression of (76) can be neglected. In addition we shall neglect the coefficient ζ in (76) as we are particularly interested in the case of a strong resonance $\zeta \ll 1$. Then a simple calculation gives for the first moment of Σ_t (36)

$$\langle \Sigma_t \rangle \cong \sum_{m \neq 0} \tilde{v}_m(j^*)e^{im\theta^*} (1 - e^{im\omega^*t}), \quad (80)$$

while the second moment can be shown to be just $\langle \Sigma_t^2 \rangle = \langle \Sigma_t \rangle^2$; hence the fidelity is, around the centre of a 2π resonance, equal to 1 within the linear response approximation

$$F(t) \cong 1, \quad \text{for } |t - kt_r| \ll \Delta_t, \quad \text{and } \zeta \ll 1. \quad (81)$$

We note that such a ‘flat-top’ structure of a 2π resonance is nicely illustrated in a numerical example in figure 3, where we consider a slightly modified model with $U_0 = \exp(-iS[\alpha(S_z/S)^2/2 + \gamma(S_z/S - j^*)^3/6])$, and $h_0(j) = \alpha j^2/2 + \gamma(j - j^*)^3/6$, such that $\omega'' = \gamma$ may not be identically vanishing.

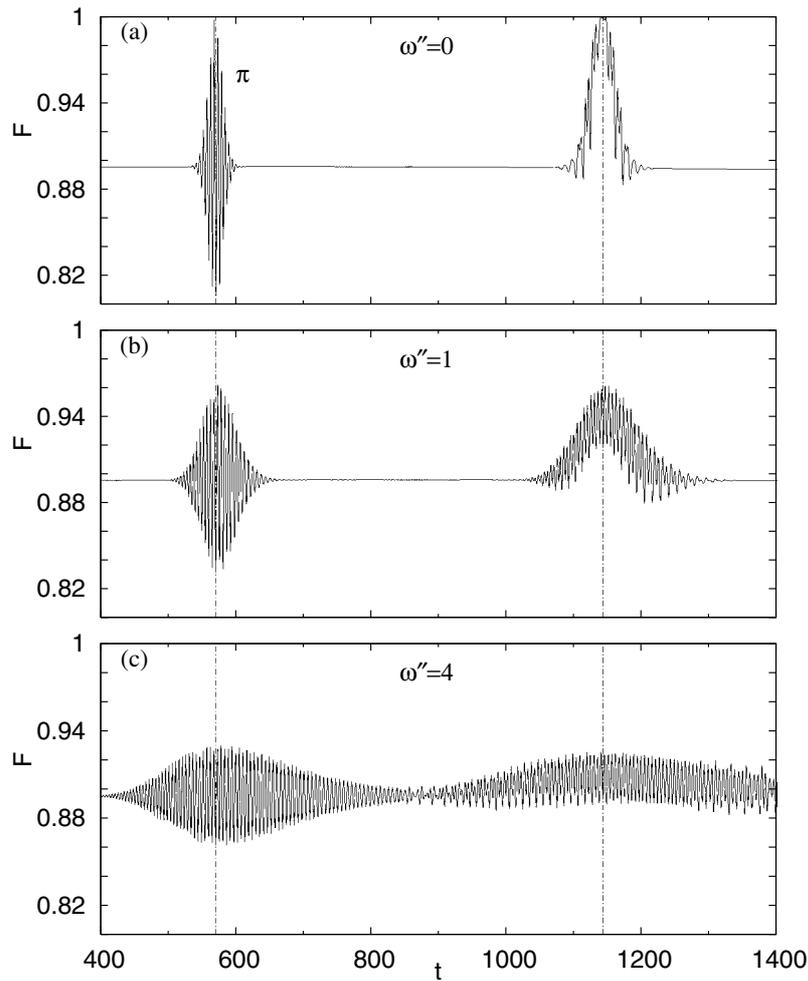


Figure 3. Structures of echo resonances for coherent initial states of the modified quantum top $U_0 = \exp(-iS[\alpha(S_z/S)^2/2 + \gamma(S_z/S - j^*)^3/6])$, $\alpha = 1.1$, $S = 200$, $\delta = 1.6 \times 10^{-3}$, for increasing value of $\omega'' = \gamma = 0$ (a), $\omega'' = 1$ (b), and $\omega'' = 4$ (c), which weakens and broadens the resonances. Note that in (a), $\omega'' = 0$, we have the same data as in figure 1(a). Vertical chain lines show theoretical times $t_r/2$ for π resonance, and t_r for 2π resonance.

5.3.2. π and $2\pi/m$ resonances. We note that one may obtain a resonance condition for time dependent term (76) with fixed m for even shorter time, namely for $t = t_r/m$. This is trivially the case for perturbations with many or at least more than one Fourier components with $|m| > 1$. However, in such cases only selected time dependent terms of the moments (36), (37) will be affected, so the fidelity will generically *not come back to 1*, even in the linear response regime (8) and in the strongly resonant case $\zeta \ll 1$. Such (incomplete) resonances at fractional times $(k/m)t_r$ will be called $2\pi/m$ resonances.

However, we may obtain a resonant condition at $t = t_r/2$ even for the first Fourier component $m = 1$ of the perturbation, due to taking the square of the operator Σ_t , thus producing Fourier number $m + m' = \pm 2$ in the last term on the RHS of equation (37). Such a resonant behaviour at times $(k + \frac{1}{2})t_r$ will be called a π resonance.

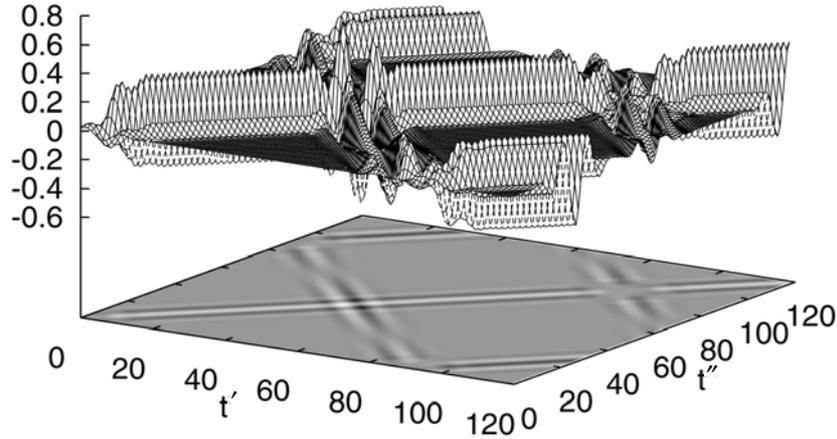


Figure 4. The two-time correlation function $C(t', t'')$, from (6), for the quantized top with $\alpha = 1.1$, $\beta = 0$, $\gamma = 0$, $S = 16$, and a coherent initial state.

So for perturbations with a *single* Fourier mode $m = \pm 1$, or more generally with only odd-numbered Fourier modes $m = 2l + 1$, the π resonance can affect only the last term of the second moment (37) and it cannot affect the first moment (36), which is given by its time-averaged value (61). To see this, we observe that the time dependent parts of form $g(\hbar n)e^{im\omega t}$ in $\langle \Sigma_t \rangle$ and $\langle \Sigma_t^2 \rangle$ are proportional to $\sum_n D_\psi(\hbar n)g(\hbar n)(-1)^{mn}$. As m is an odd number and $D_\psi(\hbar n)g(\hbar n)$ is a smooth function of n , this sum averages to zero. All this allows us to again explicitly compute the linear response fidelity (8) close to the peak in a strongly resonant case, namely

$$\langle \Sigma_t \rangle \cong \sum_{m=2l+1} \tilde{v}_m(j^*)e^{im\theta^*}, \quad \text{for } |t - (k + \frac{1}{2})t_r| \ll \Delta_t, \quad \text{and } \zeta \ll 1, \quad (82)$$

$$\langle \Sigma_t^2 \rangle \cong \left(\sum_{m=2l+1} \tilde{v}_m(j^*)e^{im\theta^*} \right)^2 + \left(\sum_{m=2l+1} \tilde{v}_m(j^*)e^{im(\theta^* + \omega^* t)} \right)^2, \quad (83)$$

$$F(t) \cong 1 - \frac{4\delta^2}{\hbar^2} \left(\sum_{m=2l+1}^{m>0} |\tilde{v}_m(j^*)| \cos(m\omega^* t + \beta_m) \right)^2, \quad (84)$$

where β_m are phases of complex numbers $\tilde{v}_m(j^*)e^{im\theta^*}$. So we have learned that the fidelity at the peak of a π resonance displays an oscillatory pattern, oscillating precisely around the plateau value F_{plateau} (63) with an amplitude of oscillations equal to $1 - F_{\text{plateau}}$ with the result that the fidelity comes back to 1 close to the peak of the resonance.

Again, our numerical example illustrates such an oscillatory structure of π resonance in figure 3. The resonances can also be nicely seen in ‘short time’ figure 1, and because $\zeta = 0$ also in the ‘long time’ figure 2. In figure 4 we depict the structure of π and 2π resonances as reflected in the two-time correlation function $C(t', t'')$. Note that the first intersection of the soliton-like trains for $t' - t'' = \text{constant}$ and $t + t' = \text{constant}$ happens at $t_r/2$ and produces a π resonance, while the second intersection at t_r produces a 2π resonance.

In analogy to the emergence of a π resonance as a consequence of the contribution from the *second* moment of Σ_t , even for the first Fourier mode $m = 1$, we shall eventually obtain also fractional $2\pi/p$ resonances at times $(k/p)t_r$ in the non-linear response regimes where higher moments $\langle \Sigma_t^p \rangle$ contribute to $F(t) \sim \langle \exp(i\Sigma_t \delta/\hbar) \rangle$, equation (45). This is illustrated numerically

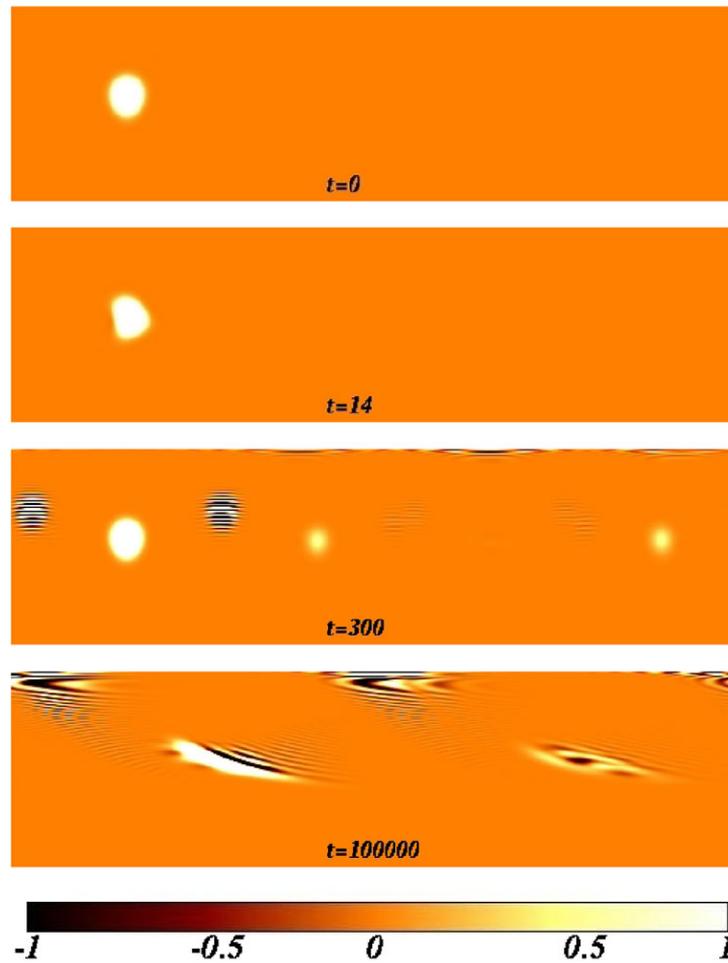


Figure 5. Movies (also online at [15].) Snapshots of the Wigner function of the echo dynamics for a quantized top, for $\alpha = 1.1$, $\beta = 0$, $\gamma = 0$ with $S = 200$ and for $\delta = 1.6 \times 10^{-3}$ (as in figures 1(a), 3(a)). The upper phase hemisphere is shown with $j = \cos \vartheta \in [0, 1]$ on the vertical axis and $\theta = \varphi \in [0, 2\pi]$ on the horizontal axis. From top to bottom we show: the initial state at $t = 0$, the state at $t = 14 \approx t_1$ when we are around the regular Ehrenfest time, at $t = 300$ in the middle of the plateau, and at $t = 100\,000$ in the ballistic regime. The bar below shows the colour code of the Wigner function values.

in figure 1(c) showing the case of strong perturbation $\delta S = 3.2$, so higher orders are important. One indeed obtains fractional resonances, some of which have been marked on the figure.

5.4. Illustration in terms of echoed Wigner functions

All the phenomena described theoretically in the preceding subsections can be nicely illustrated in terms of the *echoed Wigner function*—the Wigner function $W_{M_\delta(t)|\psi}(\mathbf{q}, \mathbf{p})$ of the echo dynamics. That is, according to formula (66) the fidelity $F(t)$ is given simply by the overlap of the echoed Wigner function and the Wigner function of the initial state. Therefore, the phase space chart of

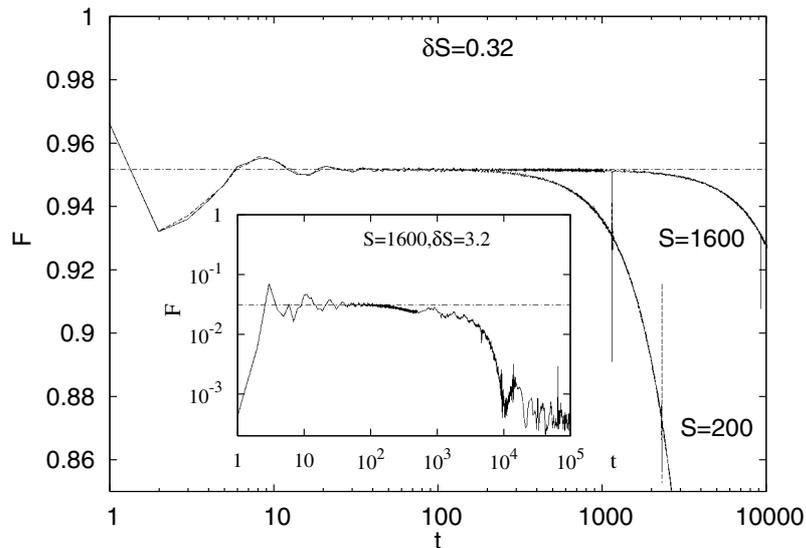


Figure 6. Short time fidelity for a quantized top with $\alpha = 1.1$, $\beta = 1.4$ and a random initial state. To reduce statistical fluctuations, averaging over 20 realizations of initial random states is performed for $S = 1600$, and over 100 initial states for $S = 200$. The horizontal chain line is the semiclassical theory (77). Resonances are present here due to the special property $\omega''(j) = 0$ and will be absent for a more generic unperturbed system. The main figure shows the case of weak perturbation $\delta S = 0.32$, whereas the inset shows the case of strong perturbation $\delta S = 3.2$.

the echoed Wigner function contains the most detailed information on the echo dynamics and illustrates the essential differences between different regimes of fidelity decay. This is shown in figure 5 (see [15]) for the quantized top where the Wigner function on a sphere is computed according to [16]. In the initial *classical regime*, $t < t_1$, the echoed Wigner function has not yet developed negative values and is in pointwise agreement with the Liouville density of the classical echo dynamics. In the *plateau regime*, $t_1 < t < t_2$, the echoed Wigner function decomposes into several pieces, one of which freezes at the position of the initial packet providing significant and constant overlap—the plateau. At very particular values of time, namely at the echo resonances, different pieces of the echoed Wigner function somehow magically recombine back into the initial state (provided that $\zeta \ll 1$). In the asymptotic, *ballistic regime*, $t > t_2$, even the frozen piece starts to drift ballistically away from the position of the initial packet, thus explaining a fast Gaussian decay of fidelity.

6. Semiclassical asymptotics: random initial state

The second specific case of interest is that of a random initial state. Here we shall assume that our Hilbert space has a finite dimension \mathcal{N} , as e.g. in the case of the kicked top or a general quantum map with a finite classical phase space, or is determined by some large classically invariant region of phase space, e.g. we may consider all states $|n\rangle$ between two energy surfaces $E_1 < h_0(\hbar n) < E_2$ of an autonomous system. In any case we have the scaling

$$\mathcal{N} \cong \frac{\mathcal{V}}{\hbar^d} \quad (85)$$

where \mathcal{V} is the classical d volume of the populated action space region of interest. The notion of a random state refers to an ensemble average over the full Hilbert space of interest. So we treat the complex coefficients ψ_n as pairs of components of a vector on a $2\mathcal{N}$ -dimensional unit sphere. In the asymptotic regime of large \mathcal{N} , these can in turn be replaced by independent complex Gaussian variables with the variance

$$\langle\langle \psi_n^* \psi_n \rangle\rangle \cong \frac{1}{\mathcal{N}} \delta_{n'n}, \quad \langle\langle \psi_n \psi_n \rangle\rangle \cong 0, \quad (86)$$

where $\langle\langle \bullet \rangle\rangle$ denotes an ensemble average over random states. When we write such an average for an operator, we actually mean

$$\langle\langle A \rangle\rangle := \langle\langle \langle \psi | A | \psi \rangle \rangle \rangle = \frac{1}{\mathcal{N}} \text{tr } A. \quad (87)$$

Note that a trivial application of the pair-contraction rule (Wick theorem) yields that averaged fidelity is asymptotically the same as the averaged fidelity amplitude squared [12]:

$$\langle\langle F(t) \rangle\rangle = \langle\langle |f(t)|^2 \rangle\rangle = |\langle\langle f(t) \rangle\rangle|^2 + \frac{1}{\mathcal{N}} \cong |\langle\langle f(t) \rangle\rangle|^2. \quad (88)$$

This means that the fidelity amplitude is self-averaging, i.e. its variance with respect to random state averaging is semiclassically vanishing. The same property holds for the fidelity itself, $\langle\langle F^2 \rangle\rangle - \langle\langle F \rangle\rangle^2 = \mathcal{O}(1/\mathcal{N})$.

6.1. The plateau: linear response and beyond

In computing the ensemble average of the linear response formula (8) we have to compute ensemble averages of the expressions (36), (37). This is a straightforward application of (86) for (37), and the pair-contraction rule for (36):

$$\begin{aligned} \langle\langle \Sigma_t^2 \rangle\rangle &\cong \frac{\tau^2}{\mathcal{N}} \sum_n \sum_{m \neq 0} |v_m(\hbar n)|^2 \frac{\sin^2(\frac{1}{2} \mathbf{m} \cdot \boldsymbol{\omega}(\hbar n) t)}{\sin^2(\frac{1}{2} \mathbf{m} \cdot \boldsymbol{\omega}(\hbar n))} \\ &\cong \frac{\tau^2}{\mathcal{V}} \int d^d \mathbf{j} \sum_{m \neq 0} |v_m(\mathbf{j})|^2 \frac{\sin^2(\frac{1}{2} \mathbf{m} \cdot \boldsymbol{\omega}(\mathbf{j}) t)}{\sin^2(\frac{1}{2} \mathbf{m} \cdot \boldsymbol{\omega}(\mathbf{j}))}, \end{aligned} \quad (89)$$

$$\langle\langle (\Sigma_t)^2 \rangle\rangle \cong \frac{1}{\mathcal{N}} \langle\langle \Sigma_t^2 \rangle\rangle. \quad (90)$$

We note that the expression (89) is just the classical phase space average $\langle \sigma_t^2 \rangle_{\text{cl}}$ where $\sigma_t(\mathbf{j}, \boldsymbol{\theta}) = \sum_{t'=0}^{t-1} v(\mathbf{j}, \boldsymbol{\theta} + \boldsymbol{\omega}(\mathbf{j}) t')$ is the classical limit of Σ_t . We see that for a random state the contribution of the square of the expectation value (90) is semiclassically small, so the plateau in fidelity is within the linear response approximation determined by the second moment (89).

On one hand, let us assume that

$$\mathbf{m} \cdot \boldsymbol{\omega}(\mathbf{j}) \neq 0 \pmod{2\pi} \quad (91)$$

for all contributing Fourier components \mathbf{m} and for all \mathbf{j} from the classical action space of interest. In such a case the \sin^2 in the denominator of (89) is never vanishing, so the integral has no problem

with singularities, while the \sin^2 in the numerator can be averaged, either over a short period of time, or over small regions of phase space for sufficiently long time $t > t_1$, in both cases yielding a trivial factor of $1/2$. In any case, the convergence timescale t_1 here is classical, and, of course, does not depend on the strength of the perturbation $t_1 \sim \hbar^0 \delta^0$. Hence in such a *non-singular* case, the linear response plateau of the fidelity reads

$$\langle\langle F(t) \rangle\rangle \cong 1 - \frac{\delta^2}{\hbar^2} \nu_{\text{ran}} + \mathcal{O}(\delta^4), \quad (92)$$

$$\nu_{\text{ran}} = \frac{\tau^2}{\mathcal{V}} \int d^d \mathbf{j} \sum_{\mathbf{m} \neq \mathbf{0}} \frac{|v_{\mathbf{m}}(\mathbf{j})|^2}{2 \sin^2(\frac{1}{2} \mathbf{m} \cdot \boldsymbol{\omega}(\mathbf{j}))}. \quad (93)$$

Going beyond the linear response approximation we again use the general formula (45), and compute the expectation value of an operator in the random state in terms of the classical phase space average

$$\langle\langle g(\mathbf{J}, \boldsymbol{\Theta}) \rangle\rangle \cong \frac{1}{(2\pi)^d \mathcal{V}} \int d^d \mathbf{j} \int d^d \boldsymbol{\theta} g(\mathbf{j}, \boldsymbol{\theta});$$

that is,

$$\langle\langle f_{\text{plateau}} \rangle\rangle \cong \frac{1}{\mathcal{V}} \int d^d \mathbf{j} \left| \int \frac{d^d \boldsymbol{\theta}}{(2\pi)^d} \exp\left(\frac{i\delta}{\hbar} \tilde{v}(\mathbf{j}, \boldsymbol{\theta})\right) \right|^2. \quad (94)$$

If, on the other hand, the condition (91) does not hold, i.e. if there exist values of the action \mathbf{j} , and $\mathbf{m} \in \mathbb{Z}^d$, $k \in \mathbb{Z}$, such that $\mathbf{m} \cdot \boldsymbol{\omega}(\mathbf{j}) = 2\pi k$, then the denominator of (89) becomes *singular* and the corresponding term (in the appropriate limit) grows in time. However, this growth stops, at least on a timescale $\sim \hbar^{-1}$, due to the discrete nature of the quantum action space. At this time the value of the correlation integral, the plateau, will be typically so small that it cannot be described within the linear response approximation, so we have to employ the general formula (45). The only modification of the general formula, with respect to a non-singular case (94), is the observation that the quantum phase space is in fact discrete in action, so one should semiclassically approximate the expectation value with the sum, instead of an integral:

$$\langle\langle g(\mathbf{J}, \boldsymbol{\Theta}) \rangle\rangle \cong \frac{1}{\mathcal{N}} \sum_{\mathbf{n}} \frac{1}{(2\pi)^d} \int d^d \boldsymbol{\theta} g(\hbar \mathbf{n}, \boldsymbol{\theta}).$$

Furthermore, the diverging terms $v_{\mathbf{m}}/\sin(\mathbf{m} \cdot \boldsymbol{\omega}/2)$ in $\tilde{v}_{\mathbf{m}}$, from (27), come from the semiclassical approximation for Σ_t whose quantum counterpart has matrix elements proportional to $V_{mn}/\sin[(\phi_m - \phi_n)/2]$. As we consider perturbations with a zero average, $V_{mm} \equiv 0$, the diverging terms are absent in the quantum operator Σ_t . Therefore to remedy the random state formula for f_{plateau} , namely (94), we simply replace an integral with the summation excluding the divergent terms, so the general final result reads

$$\langle\langle f_{\text{plateau}} \rangle\rangle \cong \frac{1}{\mathcal{N}} \sum_{\mathbf{n}}^{\mathbf{m} \cdot \boldsymbol{\omega}(\hbar \mathbf{n}) \neq 2\pi k} \left| \int \frac{d^d \boldsymbol{\theta}}{(2\pi)^d} \exp\left(\frac{i\delta}{\hbar} \tilde{v}(\hbar \mathbf{n}, \boldsymbol{\theta})\right) \right|^2. \quad (95)$$

Again we find an excellent confirmation of our theoretical predictions in the numerical experiment. In the first calculations we choose the shift $\beta = 1.4$ so that we have no singular frequency throughout the action space. In figure 6 we show the plateau, which in the case of

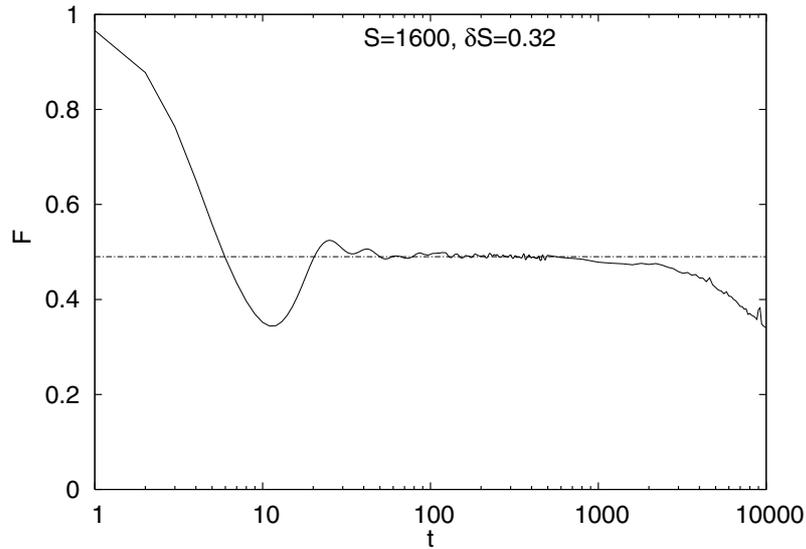


Figure 7. The short time fidelity for $\alpha = 1.1$, $\beta = 0$, $S = 1600$, $\delta S = 0.32$ and a random initial state. The chain line shows the theoretical value of the plateau as computed from formula (95).

random states starts earlier than for coherent states, namely at $t_1 \propto \hbar^0 \delta^0$. The value of the plateau can be calculated by numerically evaluating the integrals occurring in equation (93), for the linear response approximation, or equation (94) in general. The integral over the angle θ in the formula for the plateau (94) again gives a Bessel function, so we end up with a numerical integration over j :

$$\langle\langle F_{\text{plateau}} \rangle\rangle \cong \left[\frac{1}{2} \int_{-1}^1 dj J_0^2 \left(\delta S \frac{\sqrt{1-j^2}}{2 \sin\{\alpha(j-\beta)/2\}} \right) \right]^2. \quad (96)$$

Observe that the random state plateau is just a square of the action space average of a coherent state expression (67). Horizontal chain lines in figure 6 correspond to these theoretical values and agree with the numerics, both for weak perturbation $\delta S = 0.32$ and strong perturbation $\delta S = 3.2$ (inset). The plateau lasts up to t_2 which is for random states \hbar independent, $t_2 \sim 1/\delta$. Small resonances visible in the figures are due to the fact that the Hamiltonian is a quadratic function of the action and therefore $\omega'' \equiv 0$, so the resonance condition (74) is satisfied also for extended states (77). For a more generic Hamiltonian these narrow resonant spikes will be absent. In figure 7 we also demonstrate the plateau in the fidelity for the zero-shift case $\beta = 0$ with a singular frequency, $\omega(j=0) = 0$, where we again find an excellent agreement with the theoretical prediction (95). In this case the theoretical value has been obtained by replacing an integral in (96) with a sum over n (replacing $j = \hbar n$) and summing over all quantum numbers except $n = 0$. Observe that the value of the plateau is much lower than in the case of a non-zero shift, $\beta = 1.4$, in figure 6.

6.2. Asymptotic regime of long times

Again, after sufficiently long time $t_2 \sim \delta^{-1}$, the second term in BCH expansion (17) will be dominant and we shall use the ensemble average of the ASI formula (31) where $\langle\langle D_\psi \rangle\rangle = \frac{1}{N}$:

$$\langle\langle f(t) \rangle\rangle \cong \frac{1}{\mathcal{V}} \int d^d j \exp\left(i \frac{\tau \delta^2}{2\hbar} \bar{v}(j) t\right). \quad (97)$$

The semiclassical computation of such an integral is an elementary application of the stationary phase method in d dimensions, following [6] for the analogous case of a generic observable. The condition for the validity of the stationary phase method is that $t\delta^2/\hbar > 1$, which will turn out to be consistent with the assumption that $t > t_2$. Let \mathbf{j}_η , $\eta = 1, \dots, p$, be the p points where the phase of the exponential on RHS of (97) is stationary, $\partial \bar{v}(\mathbf{j}_\eta)/\partial \mathbf{j} = 0$. This yields a simple result:

$$\langle\langle f(t) \rangle\rangle \cong \frac{(2\pi)^{3d/2}}{\mathcal{V}} \left| \frac{\hbar}{t\tau\delta^2} \right|^{d/2} \sum_{\eta=1}^p \frac{\exp\{i\bar{v}(\mathbf{j}_\eta)t\tau\delta^2/(2\hbar) + i\nu_\eta\}}{|\det W_\eta|^{1/2}}, \quad (98)$$

where

$$(W_\eta)_{jk} := \frac{\partial^2 \bar{v}(\mathbf{j}_\eta)}{\partial j_k \partial j_l} \quad (99)$$

is a matrix of second derivatives at the stationary point η , and $\nu_\eta = \pi(m_+ - m_-)/4$ where m_\pm are numbers of positive/negative eigenvalues of the matrix W_η . Here we should remember that the asymptotic formula (97) has been obtained as a stationary phase approximation of an integral in the limit of an infinite action space. If we have a finite region of the action space, the stationary phase approximation of (97) gives an additional oscillating prefactor, whose amplitude dies out as $(\hbar/t)^{1/2}$ for $\hbar \rightarrow 0$ and/or $t \rightarrow \infty$, and which can be interpreted as a *diffraction*. This oscillating prefactor can be seen in numerical data for fidelity in the inset of figure 8.

So we have found that, apart from possible oscillation due to phase differences if $p > 1$, the fidelity will for a random state asymptotically decrease with a power law

$$\langle\langle F(t) \rangle\rangle \sim \left(\frac{t}{t_{\text{ran}}}\right)^{-d}, \quad t_{\text{ran}} = \text{constant} \times \frac{\hbar}{\delta^2}. \quad (100)$$

Note that for a random initial state, the actual transition time scale t_2 , as predicted by equation (30), is indeed independent of \hbar .

The above theory is again quantitatively confirmed in figure 8 for the quantized top with $\alpha = 1.1$, $\beta = 0$, where a (single) stationary point needed for the formula (98) had to be calculated numerically.

7. Discussion

In the present paper we have elaborated a semiclassical theory of quantum fidelity decay for systems with integrable classical counterparts, perturbed by observables of vanishing time average. Such perturbations may not be generic, but provide an important special class of perturbations which are often enforced by symmetries.

We have found that quantum fidelity will generally, after initial decay on a short perturbation independent timescale t_1 , exhibit a saturation around a constant value—the plateau, and stay there up to time t_2 , such that the time span of the plateau $t_2/t_1 \sim 1/\delta$ can be made arbitrarily long for small perturbation δ . After the plateau, $t > t_2$, the fidelity will decay as a Gaussian for a coherent initial state, or as a power law t^{-d} for random initial states, just to name the two most important specific cases, where the timescale of the decay is generally proportional to δ^{-2} . This

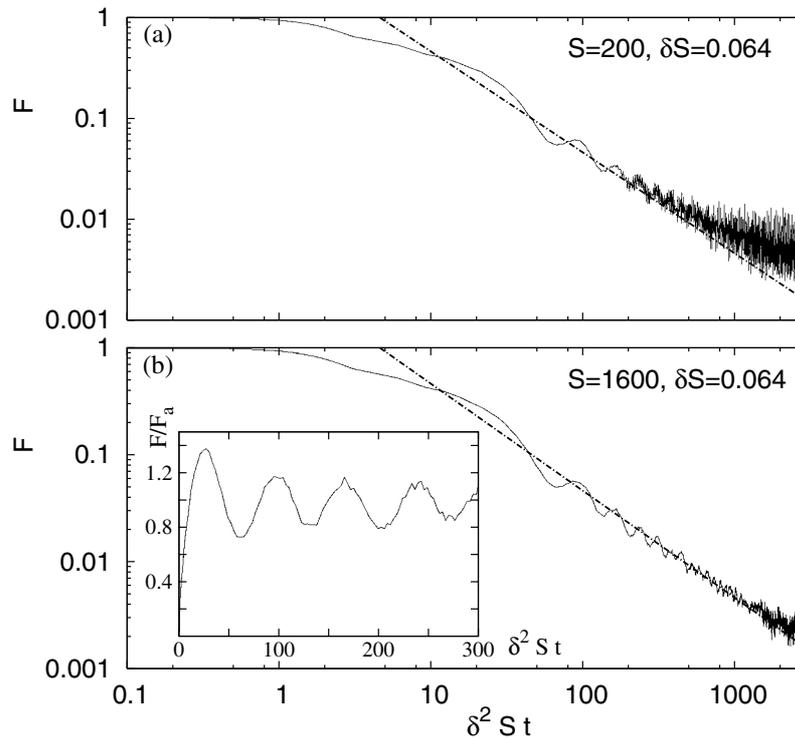


Figure 8. The long time fidelity for random states for a quantized top with $\alpha = 1.1, \beta = 0$, for $S = 200$ (a) and $S = 1600$ (b). Here $\delta S = 0.064$ and averaging for $S = 200$ and $S = 1600$ is performed over 1000 and 20 initial random states, respectively. The heavy chain line shows the theoretical asymptotic decay (97) with an analytically computed prefactor (no fitting parameters). The inset in the bottom figure shows the diffractive quotient between the numerical fidelity and the asymptotic formula (98) (the chain line in the main figure).

must be contrasted with the decay in the regular case of a non-zero time-average perturbation [6] where the decay time scales with the perturbation strength as δ^{-1} . In the case of localized initial wavepackets in one dimension, we observe and explain the effect of echo resonances, i.e. the sudden revivals of fidelity at perturbation independent and equally spaced instants of time.

The freezing of fidelity is a distinct quantum phenomenon, as the corresponding classical fidelity for the initial Gaussian wavepacket displays a power law decay t^{-d} [8] after the point t_1 where the quantum plateau starts. The classical fidelity decays on a timescale δ^{-1} no matter what the average value of the perturbation is, while the timescale of quantum fidelity decay drastically changes from δ^{-1} to δ^{-2} , having $\bar{V} = 0$. This increased stability of regular quantum systems to perturbations with a zero time average could be potentially useful in constructing quantum devices [17]—even more so because the plateau also exists for random initial states which are expected to be more relevant for efficient quantum information processing.

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