J. Phys. A: Math. Gen. 27 (1994) 8059–8077. Printed in the UK

Semiclassical energy level statistics in the transition region between integrability and chaos: transition from Brody-like to Berry–Robnik behaviour

Tomaž Prosen† and Marko Robnik‡

Center for Applied Mathematics and Theoretical Physics, University of Maribor, Krekova 2, SLO-62000 Maribor, Slovenia

Received 23 June 1994, in final form 16 September 1994

Abstract. We study the energy level statistics of the generic Hamiltonian systems in the transition region between integrability and chaos and present the theoretical and numerical evidence that in the ultimate (far) semiclassical limit the Berry–Robnik (1984) approach is the asymptotically exact theory. However, before reaching that limit, one observes phenomenologically a quasi-universal behaviour characterized by the fractional power-law level repulsion and globally quite well described by the Brody (or Izrailev) distribution. We offer theoretical arguments explaining this extremely slow transition and demonstrate it numerically in improved statistics of the Robnik billiard and in the standard (Chirikov) map on a torus.

1. Introduction

In this paper we do not intend to offer a complete review of the fundamentals of quantum chaos but rather spend just a few introductory words making the paper self-contained. In the development of quantum chaos much understanding has been achieved by studying the statistical properties of the (quasi-)energy spectra (and of other observables) in quantum systems whose classical counterparts are non-integrable and chaotic. For recent reviews see papers in Giannoni *et al* (1991), Haake (1991), Gutzwiller's book (1990), Eckhardt (1988), Bohigas and Giannoni (1984) and Robnik (1994). We know that there are three universality classes of spectral fluctuations: Poisson statistics in the classically integrable cases; in the case of classical ergodicity we find the GOE/GUE statistics of random-matrix theories depending on whether there is one/none anti-unitary symmetry (we ignore spin). The interesting and difficult case of mixed-type classical dynamics of KAM-like (generic) systems has been studied numerically for the first time by Robnik (1984), where a continuous transition from Poisson to GOE statistics in a billiard system (Robnik 1983) was found, and this work has been substantially revised in Prosen and Robnik (1993a). Further theoretical progress was published by Berry and Robnik (1984) where the following semiclassical theory of the level spacings was presented. The eigenstates (their Wigner functions in phase space) are supposed to condense uniformly on the underlying classical invariant regions such that each of them—in the semiclassical limit—supports a level sequence which, for itself, has Poisson or GOE statistics if the region is regular or irregular, respectively. All

† E-mail address: Tomaz.Prosen@UNI-MB.SI
‡ E-mail address: Robnik@UNI-MB.SI

the regular regions can be thought of as supporting a single Poisson sequence because the Poisson statistics is preserved upon a statistically independent superposition. The mean-level spacing of such a sequence is determined by the fractional phase-space volume of the regular regions. On the other hand each chaotic (GOE) level sequence has a mean-level spacing governed by the corresponding fractional phase-space volume. The entire spectrum is then assumed to be a statistically independent superposition of all subsequences. The statistical independence in the semiclassical limit is justified by the principle of uniform semiclassical condensation of eigenstates (in the phase space) and by the lack of their mutual overlap, consistent with Percival's (1973) conjecture. Thus the problem of the statistics of the entire spectrum is now precisely formulated mathematically (this forms the essence of the Berry–Robnik approach) and its solution as far as the level spacings are concerned can be expressed in the following way: the statistical independence of superposition implies factorization of the gap distribution functions (Mehta 1991, Haake 1991): the probability that there is no level within a gap clearly factorizes upon a statistically independent superposition. The connection between the level-spacing distribution $P(S)$ and the gap distribution $E(S)$ is as follows:

$$P(S) = \frac{\mathrm{d}^2 E(S)}{\mathrm{d}S^2} \tag{1}$$

and conversely

$$E(S) = \int_S^\infty \mathrm{d}\sigma \; (\sigma - S) P(\sigma) \,. \tag{2}$$

Leaving aside the general case of infinitely many chaotic components, which does not include anything surprisingly new, let us restrict ourselves to the case of one regular component with mean-level density ρ_1 (= fractional phase-space volume) and one chaotic component with the mean-level density ρ_2 where $\rho_1 + \rho_2 = 1$. This is going to be already an excellent approximation because in a generic system of a mixed type there is usually only one large and dominating chaotic region. Following Mehta (1991), Haake (1991) and Berry and Robnik (1984) we have

$$E(S) = E_{\mathrm{Poisson}}(\rho_1 S) E_{\mathrm{GOE}}(\rho_2 S) \tag{3}$$

where the Poissonian gap distribution E_{Poisson} is

$$E_{\mathrm{Poisson}}(S) = \exp(-S) \tag{4}$$

whereas for the E_{GOE} there is no simple closed formula (for the infinitely-dimensional GOE case) and must be worked out by using practical approximations for P_{GOE} and/or E_{GOE} which e.g. can be found in Haake (1991), pp 72–4. However the two-dimensional GOE case (the so-called Wigner surmise) can be worked out explicitly as given in Berry and Robnik (1984), formula (28), which is usually a good starting approximation.

As for the delta statistics $\Delta(L)$ a similar procedure based on the assumption of statistical independence leads to the simple (additive) formula (Seligman and Verbaarschot 1985)

$$\Delta(L) = \Delta_{\mathrm{Poisson}}(\rho_1 L) + \Delta_{\mathrm{GOE}}(\rho_2 L) \tag{5}$$

where $\Delta_{\mathrm{Poisson}}(L) = L/15$ whilst for Δ_{GOE} there are good approximations given in Bohigas (1991).

The main objective of this paper is to clearly demonstrate and explain the Berry–Robnik regime. However, before such a regime is formed in the ultimate far-semiclassical limit we typically observe a quasi-universal behaviour in the spectral statistics which is characterized by the fractional power-law level repulsion and globally the adequacy of the

Brody (1973, 1981) distribution and of similar distributions such as Izrailev (1989). A thorough numerical study of this phenomenon has been published recently by Prosen and Robnik (1993a), henceforth referred to as PR, and also Ganesan and Lakshmanan (1994) and previously by Seligman *et al* (1984), Wintgen and Friedrich (1987), Hönig and Wintgen (1989) and Meyer *et al* (1984) (see also Meyer 1986). So, another goal of this paper is to present additional (possibly improved) numerical evidence (even somewhat better than in PR) for the existence of the fractional power-law level repulsion. This is given in the next section. In section 3 we present the theoretical arguments explaining this quasi-universality. In section 4 we show the transition from the Brody-like towards the Berry–Robnik regime in our billiard system. In section 5 we study the same aspects in the quantized standard map on a torus, but also present clear evidence for the establishment of the Berry–Robnik regime. In section 6 we review the ingredients concerning the structure of eigenfunctions which underlie the Berry–Robnik theory. In section 7 we discuss the main results and draw general conclusions.

2. Revised statistics of a billiard system: the evidence for the fractional power-law level repulsion in the near-semiclassical regime

The billiard system analysed in this section has been introduced by Robnik (1983) and is defined as the quadratic conformal map of the unit disk $|z| \leqslant 1$ in the z-plane onto the w-plane, namely

$$w = z + \lambda z^2 . \tag{6}$$

At $\lambda = 0$ we have the integrable circular billiard, for $0 < \lambda < \frac{1}{4}$ the billiard is convex and since it has an analytic boundary the KAM theory applies, and is thus a truly generic system unlike some other frequently studied billiards such as the Sinai billiard, or the stadium of the Bunimovich and Benettin–Strelcyn billiard. It has mixed dynamics in phase space. For $\lambda > \frac{1}{4}$ the shape is non-convex so that Lazutkin caustics and the related invariant tori no longer exist (even for $\lambda = \frac{1}{4}$ this has been rigorously proven by Mather (1982)), thus enabling ergodicity. However Hayli *et al* (1987) have shown that there exists a family of stable periodic orbits (surrounded by very tiny stability islands) which undergoes a cascade of period-doubling bifurcations with the estimated and extrapolated limiting value of $\lambda = 0.279\ldots$. On the other hand Markarian (1993) has shown rigorously that at $\lambda = \frac{1}{2}$ (cardioid billiard, having a cusp singularity) the system is ergodic, mixing and K-system. Li and Robnik (1994c) have strong numerical evidence for ergodicity at $\lambda = \frac{3}{8}$. The system has also been studied extensively by Frisk (1990), Stone and Bruus (1993, 1994) and Bruus and Stone (1994).

The quantum mechanics of this billiard problem is embodied in the following eigenvalue problem:

$$\Delta\psi + E\psi = 0 \qquad \psi = 0 \quad \text{on the boundary} . \tag{7}$$

To solve this Schrödinger eigenvalue problem for the eigenfunctions ψ and the eigenvalues (energies) E we have employed the powerful conformal mapping diagonalization technique described in detail in PR. In order to study the spectral fluctuation properties we have to separate the mean smooth part from the fluctuating part of the spectral staircase function (the so-called unfolding procedure) where we use the well known Weyl formula with perimeter and curvature corrections. After performing this unfolding procedure (the spectral staircase now has by construction a unit mean-level spacing) we begin with the statistical

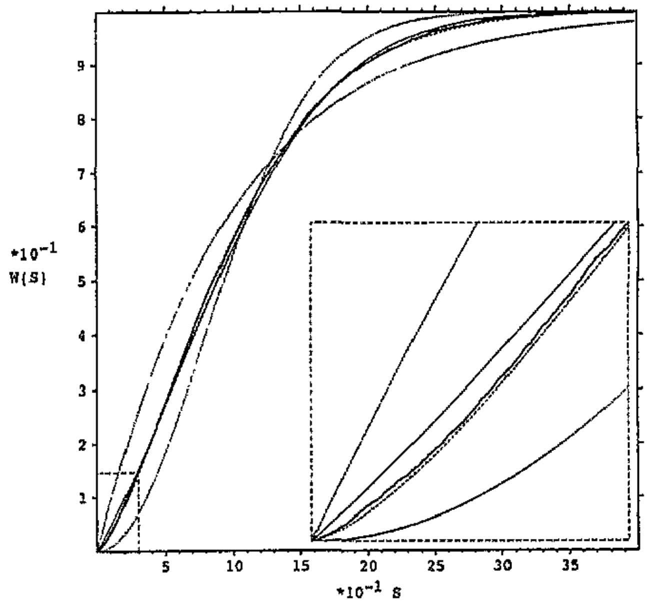

Figure 1. The figure shows the cumulative level-spacing distribution for the mixed regime of our billiard system $\lambda = 0.175$ and for 11 637 consecutive levels of both parities in the energy range $88\,000 < E < 132\,000$. The dotted curves represent the limiting Poisson and GOE statistics. The thin full curve is the best-fit Berry–Robnik distribution and the broken curve is the best-fit Brody distribution while the thicker full curve displays the numerical data. Although we have used the highest energy data that we were able to calculate one can still see that the numerical distribution is still much closer to Brody with $\beta = 0.43$ than to the limiting semiclassical Berry–Robnik distribution.

analysis of the spectral fluctuations. Since we are interested predominantly in short-range spectral correlations we only show the results for (nearest-neighbour) energy-level spacings. The focus of our attention is the phenomenon of the fractional power-law level repulsion which is characterized by the locally defined exponent β whose value in practice is, however, determined by a global fit with a suitable one-parameter family of level-spacing distributions. One such distribution which is sufficient for our purposes is the well known Brody distribution whose cumulative form reads

$$W^{\mathrm{B}}(S,\beta) = 1 - \exp\left(-bS^{\beta+1}\right) \qquad b = \{\Gamma((\beta+2)/(\beta+1))\}^{\beta+1} \,. \tag{8}$$

It turns out quite generally that the presentation and statistical analysis of cumulative distributions has many advantages. In figure 1 we show the numerical cumulative level-spacing distribution $W(S)$ for $\lambda = 0.175$ which should be compared with the best-fit Brody distribution (8) with $\beta = 0.43$. Here, we are obviously very deep in the Brody-like regime of fractional power-law level repulsion and still very far from the best fitting semiclassical limiting Berry–Robnik distribution.

In this spirit while working with $W(S)$ and following PR, we define the so-called T-*function* which is the following transformation of the spectral cumulative level-spacing distribution $W(S) = \int_0^S \mathrm{d}t\, P(t)$ (where $P(S)$ is the level-spacing distribution after unfolding),

$$T(\sigma) = \ln(-\ln(1 - W(\exp\sigma))) \qquad \sigma = \ln S \,. \tag{9}$$

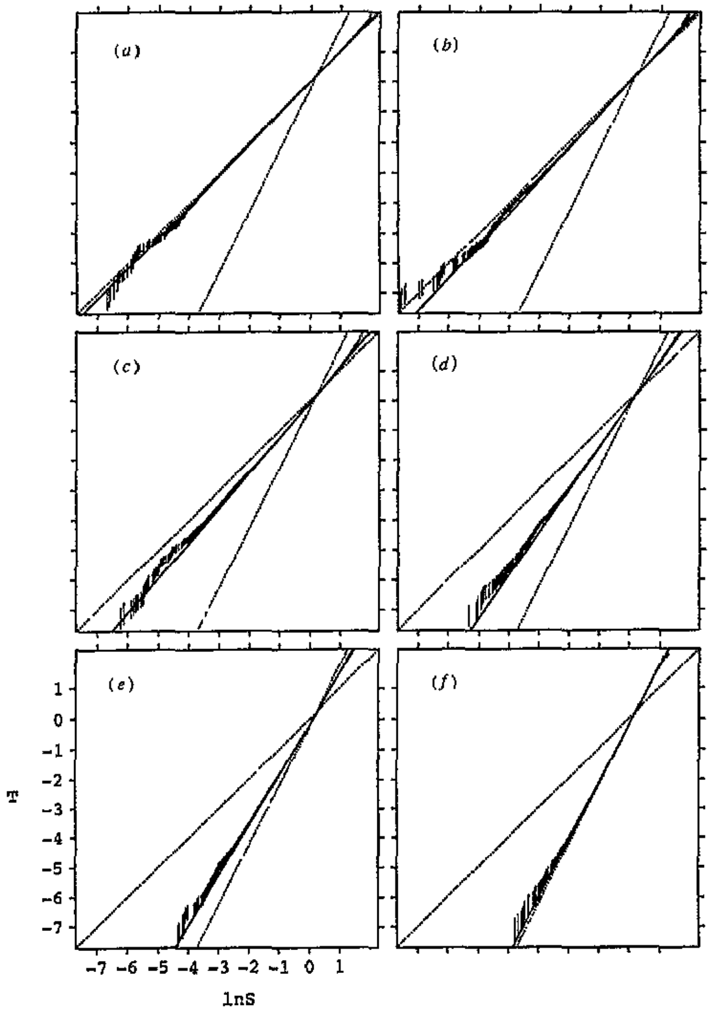

Figure 2. In this figure we illustrate the phenomenon of power-law level repulsion by means of the T-function of the level-spacing distribution for the transition from integrability to chaos in our billiard system, (*a*) $\lambda = 0.100$, (*b*) 0.125, (*c*) 0.150, (*d*) 0.175, (*e*) 0.200 and (*f*) 0.375. The numerical data are plotted in the form of plus/minus one-sigma bars. The straight lines represent the limiting Poisson and GOE statistics (dotted) and the best-fit Brody statistics (full). The energy intervals that were used are the same as in table 1.

This unusual transformation has the property that it transforms the Brody distribution (8) to the straight line

$$T_{\beta}^{\mathrm{B}}(\sigma) = (\beta + 1)\sigma + \ln b\,. \qquad (10)$$

In a graphical presentation the slope of the best-fit line minus one gives the so-called *level-repulsion exponent* β. By using this method (whose detailed exposition is given in PR) we have analysed the energy spectra of our billiard system at six different values of the shape parameter $\lambda = 0.1$, 0.125, 0.15, 0.175, 0.2 and 0.375, as shown in figure 2. We have been able to calculate 45 000 accurate energy levels (of both parities) for $\lambda = 0.1$ and 25 000 accurate energy levels at $\lambda = 0.375$. In order to make the effective value of Planck's

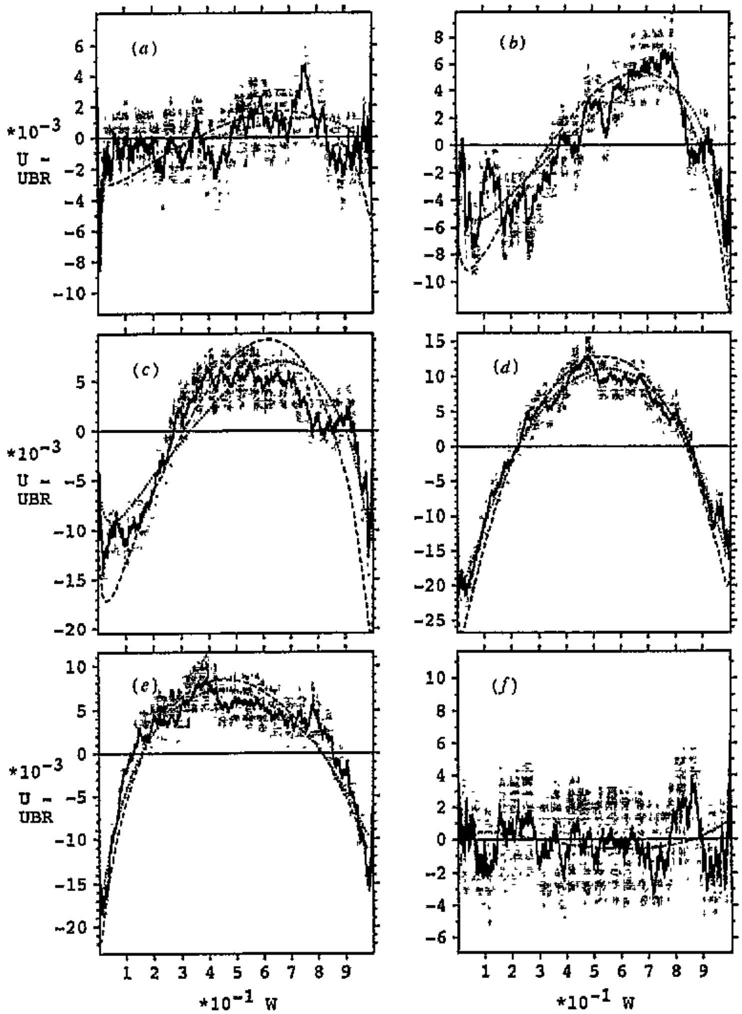

Figure 3. This is the most detailed and uniform representation of the level-spacing distribution by means of the U-function: we plot its deviation from the U-function of the best-fit limiting Berry–Robnik distribution $U(W(S)) - U(W_2^{\mathrm{BR}(\infty)}(S, \rho_1))$ versus $W(S)$ for the transition from integrability to chaos in our billiard system, (*a*) $\lambda = 0.100$, (*b*) 0.125, (*c*) 0.150, (*d*) 0.175, (*e*) 0.200 and (*f*) 0.375. The numerical data (noisy curves inside $\pm 1\sigma$ grey band) are also compared with the deviation of the best-fit Brody distribution (broken curves) and with the deviation of the best-fit BRB distribution (dotted curves). The energy intervals that were used are the same as in table 1.

constant $\hbar_{\mathrm{eff}} = 1/\sqrt{E}$ well defined we have discarded the lower two thirds of our spectra so the presented results are obtained using only the upper third of the energy levels. The general conclusion is that the fractional power-law level repulsion exists and extends over several orders of magnitude in S.

The T-function representation nicely describes this phenomenon but has the disadvantage of being biased to larger values of S, where the expected and the actual statistical error are smaller than at smaller values of S. This difficulty is removed in another representation namely the so-called *U-function* (transformation of $W(S)$),

$$U(W) = (2/\pi)\cos^{-1}\sqrt{1-W} \tag{11}$$

Table 1. The table includes the values of all the relevant parameters which were obtained by the least-squares fit of the level-spacing distribution with the three theoretical distributions: Berry–Robnik (columns 3, 4), Brody (columns 5, 6), and combined BRB (columns 7–9) covering the transition from integrability to chaos in our billiard system with the six chosen values of deformation parameter λ. N is the number of consecutive levels which were used for statistical analysis starting at $\approx 2N$th and ending at $\approx 3N$th consecutive level of each parity. The quality of the fit is characterized by the value of the ratio χ^2/N which ideally should be less than one. In the 10th column we give the measure of the classical regular regions.

λ	N	β^{B}	χ^2/N	ρ_1^{BR}	χ^2/N	β^{BRB}	ρ_1^{BRB}	χ^2/N	ρ_1^{cl}
0.100	13733	0.02	0.3	0.80	0.4	0.32	0.73	0.2	0.88
0.125	12861	0.08	0.9	0.65	2.0	0.28	0.47	0.4	0.70
0.150	12239	0.17	2.0	0.49	4.2	0.41	0.33	0.7	0.36
0.175	11637	0.43	0.8	0.25	10.5	0.52	0.08	0.2	0.17
0.200	11037	0.69	0.5	0.11	4.7	0.72	0.02	0.3	0.05
0.375	7977	0.94	0.2	0.007	0.2	0.94	0.000	0.2	0.00

where the estimated statistical error is now $\delta U = 1/(\pi\sqrt{N})$, N being the number of numerical spacings, which is constant at all S. In figure 3 we show the difference between the actual U-function $U(W(S))$ and the best-fit limiting Berry–Robnik (based on exact infinite-dimentional GOE expressions for $E_{\mathrm{GOE}}(S)$ in (3), rather than the Wigner surmise (two-dimentional GOE)) $U^{\mathrm{BR}} = U(W_2^{\mathrm{BR}(\infty)}(S, \rho_1))$ versus $W(S)$. In graphs of this type the density of objects on the abscissa is constant. In these plots the chosen values of λ are the same as in figure 2.

In table 1 we give the numerical values of the relevant parameters and statistical measures for three distributions including the Berry–Robnik–Brody distribution introduced in the next section.

3. Theoretical arguments explaining the quasi-universal Brody-like regime: the fractional power-law level repulsion

Since the fractional power-law level repulsion is a quasi-universal and very robust phenomenon it is not surprising that it can be explained in terms of very general arguments. In fact the only important relevant ingredient is the validity of the semiclassical (action) tunnelling formula for the size of avoided crossings. This formula, which we will introduce below, is certainly correct in integrable systems, but is expected to also be correct in cases of soft KAM chaos as shown by Wilkinson (1987), and might fail in cases of hard KAM chaos as demonstrated by Bohigas *et al* (1993).

Let us take certain quantum KAM Hamiltonian $\widehat{H}$ and two states $|\psi_1\rangle$ and $|\psi_2\rangle$ whose phase-space distributions (Wigner or Husimi) are localized on the disjoint classical invariant components, e.g. on KAM tori separated by a broken separatrix. Of course, $|\psi_1\rangle$ and $|\psi_2\rangle$ are typically not exact eigenstates of $\widehat{H}$, but we shall assume that the expected energies $E_j = \langle\psi_j|\widehat{H}|\psi_j\rangle$ are close together (much closer than the mean-level spacing) so that the true eigenstates $|\psi\rangle$ can be calculated inside this 2×2 model, written as linear combinations of these non-communicating disjoint states

$$\sum_{k=1}^{2}(\langle\psi_j|\widehat{H}|\psi_k\rangle c_k - E\langle\psi_j|\psi_k\rangle) = 0 \qquad |\psi\rangle = \sum_{k=1}^{2} c_k|\psi_k\rangle\,. \tag{12}$$

The true eigenenergies are simple solutions of the quadratic secular equation whose difference, the level spacing S reads

$$S = \frac{1}{1-O_{12}^2}\sqrt{(1-O_{12}^2)S_0^2 + 4(H_{12}-\bar{E}O_{12})^2} \tag{13}$$

where S_0 is the energy spacing of the disjoint states $S_0 = |E_2 - E_1|$ and $\bar{E}$ is an average energy $\bar{E} = (E_1 + E_2)/2$. H_{12} and O_{12} are the overlaps $H_{12} = \langle\psi_1|\widehat{H}|\psi_2\rangle$, $O_{12} = \langle\psi_1|\psi_2\rangle$ which are assumed to be small ($\mathcal{O}(\exp(-\text{constant}/\hbar))$), since we know that the states ψ_1 and ψ_2 are exponentially localized on the disjoint classical invariant sets, and so one may write

$$S = \sqrt{S_0^2+\delta^2}\left(1+\mathcal{O}(O_{12})^2\right) \qquad \delta = 2(H_{12}-\bar{E}O_{12}). \tag{14}$$

Let us now consider the statistics of the ensemble of such spacings S. Uncorrelated spacing S_0 and the overlap δ (which can also be interpreted as the size of the avoided crossing) can be considered as the two, obviously independent, random variables. The first random variable S_0 is obviously distributed according to the Poisson distribution

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}S_0} = \mathrm{e}^{-S_0} \tag{15}$$

(where we choose the energy units so that $\langle S_0\rangle = 1$), since the expectation values of the Hamiltonian $E_{1,2}$ for states having Wigner phase-space distribution functions with disjoint supports cannot be correlated. The second random variable δ can be written in terms of another random variable, namely the classical tunneling action η (Wilkinson 1987)

$$\delta = \mathrm{e}^{-\eta/\hbar} \tag{16}$$

where we absorb the non-essential prefactor into the definition of η which makes a negligible shift of order $\mathcal{O}(\hbar\ln\hbar)$ when $\hbar\to 0$. The probability distribution of tunnelling actions $\phi(\eta) = \mathrm{d}\mathcal{P}/\mathrm{d}\eta$ depends solely on classical dynamics and is independent of the $\hbar$ (at least asymptotically as $\hbar\to 0$). We calculate the level-spacing distribution $P(S) = \mathrm{d}\mathcal{P}/\mathrm{d}S$ in two steps.

(i) We calculate it for fixed value of δ

$$P(S)|_\delta = \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}S_0}\left.\frac{\mathrm{d}S_0}{\mathrm{d}S}\right|_\delta = \frac{S\theta(S^2-\delta^2)}{\sqrt{S^2-\delta^2}}\mathrm{e}^{-\sqrt{S^2-\delta^2}}. \tag{17}$$

(ii) Then we integrate over δ weighted with its probability distribution

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\delta} = \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\eta}\left|\frac{\mathrm{d}\eta}{\mathrm{d}\delta}\right| = \frac{\hbar}{\delta}\phi(-\hbar\ln\delta) \tag{18}$$

giving

$$P(S) = \int_0^\infty \mathrm{d}\delta\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\delta}P(S)|_\delta = \int_0^\infty \mathrm{d}x\frac{\phi(x-\hbar\ln S)}{\sqrt{1-\exp(-2x/\hbar)}} \tag{19}$$

where we have introduced a new integration variable $x = -\hbar\ln(\delta/S)$ and used an approximation $\mathrm{e}^{-\sqrt{S^2-\delta^2}}\approx 1$ since we are only interested in the small spacing region $S \ll 1$, where the phenomenon of level repulsion exists. Now the power-law level-repulsion phenomenon can be shown to be a simple consequence of the fact that $P(S)$ can be written as a power series in $\ln S$ rather than in S provided that $\phi(x)$ is nice enough. When approaching semiclassical limit $\hbar\to 0$, it is enough to require that the tunnelling action distribution $\phi(x)$ is smooth and differentiable and expand $\phi(x-\hbar\ln S)$ into the first-order Taylor series around x

$$P(S) \approx \int_0^\infty \mathrm{d}x\frac{\phi(x)-\hbar\ln S\phi'(x)}{\sqrt{1-\exp(-2x/\hbar)}} = c_0 - \hbar\ln Sc_1 \approx c_0\mathrm{e}^{-\hbar c_1\ln S/c_0} = c_0S^{-\hbar c_1/c_0} \tag{20}$$

if $|\hbar \ln Sc_1/c_0| \ll 1$ The coefficients c_0, c_1 can we written systematically as power series in $\hbar$ but we are satisfied with the first order

$$c_0 = \int_0^\infty \mathrm{d}x \frac{\phi(x)}{\sqrt{1-\exp(-2x/\hbar)}} = 1 + \mathcal{O}(\hbar) \tag{21}$$

$$c_1 = \int_0^\infty \mathrm{d}x \frac{\phi'(x)}{\sqrt{1-\exp(-2x/\hbar)}} = -\phi(0) + \mathcal{O}(\hbar) \,. \tag{22}$$

Thus we have shown that the phenomenon of the fractional power-law level repulsion exists on the interval of S

$$P(S) \propto S^\beta \qquad \mathrm{e}^{-1/\beta} < S < 1 \tag{23}$$

where the level-repulsion exponent goes to zero linearly as $\hbar \to 0$

$$\beta = \hbar\phi(0) + \mathcal{O}(\hbar^2) \tag{24}$$

which is compatible with the validity of the Berry–Robnik formula which does not exhibit level repulsion, $P^{\mathrm{BR}}(0) \neq 0 \ \Rightarrow \ \beta = 0$, in the strict semiclassical limit.

As explained at the beginning of this section the existence of the fractional power-law level-repulsion embodied in (23) and (24) depends crucially upon the validity of the semiclassical (action) tunnelling formula (16), which might fail in chaotic systems as shown by Bohigas *et al* (1993) even if the two supporting classical chaotic components are disjoint. However, if for some reason (e.g. diffusively localized classical dynamics in a chaotic region) the quantal eigenstates as described by the Wigner functions are strongly localized, then they might mimic two regular states and (16) might be satisfied approximately implying the existence of the fractional power-law level repulsion and globally Brody-like behaviour of the corresponding sequence of purely *irregular* levels. Thus we can allow qualitatively for the Brody-like regime in a completely chaotic system, but the dependence on $\hbar$ in this case is much more complicated and the linear dependence of (24) on $\hbar$ does not apply: indeed, as $\hbar \to 0$ we have to recover $\beta = 1$ and not $\beta = 0$. Such a scenario might be closely related to the analysis of localization of chaotic eigenstates in time-dependent systems and its relevance for the Brody-like behaviour of the quasi-energy spectra. See, for example, Chirikov (1991) and Casati and Chirikov (1994) and the references therein.

But once again we should explicitly emphasize that in this section we are not deriving the Brody distribution which still has no physical foundation whatsoever, but only the fractional power-law level repulsion which is just a local property of $P(S)$ at small S. Its globalization results in what we carefully call a Brody-like distribution which, for example, could be very well described or approximated by the Izrailev distribution (Izrailev 1989) or other similar formulae. For the sake of simplicity and in view of our ignorance of the precise global features of such a distribution we have used just the Brody-distribution as the mathematically simplest one which certainly correctly captures the small-S behaviour.

Of course, generally there is no procedure for a separation of regular and irregular levels in a generic conservative system unless the regular levels are characterized by the almost-degeneracy implied by some discrete symmetry (Bohigas *et al* 1993). Therefore it is difficult to test the above ideas in a mixed system, whilst in an ergodic system we might be more lucky. Under such circumstances for the purpose of purely phenomenological analysis of the Brody-to-Berry–Robnik transition in a mixed system we propose to use a two-parameter level-spacing distribution. The regular levels with measure (density of levels) ρ_1 are supposed to obey Poisson statistics, and the irregular levels with measure $\rho_2 = 1 - \rho_1$ are supposed to obey the Brody distribution with some β which is supposed to capture the localization of the underlying chaotic states. Thus assuming statistical independence of regular and irregular levels we have introduced a two-parameter level statistics with gap distribution

$$E^{\mathrm{BRB}}(S,\rho_1,\beta) = E_{\mathrm{Poisson}}(\rho_1 S)E^{\mathrm{B}}(\rho_2 S,\beta) = \mathrm{e}^{-\rho_1 S}E^{\mathrm{B}}((1-\rho_1)S,\beta) \tag{25}$$

where Brody statistics has a gap distribution which can be expressed in terms of an incomplete gamma function Q (see, for example, Abramowitz and Stegun 1965)

$$E^{\mathrm{B}}(S,\beta) = Q\left(\frac{1}{\beta+1}, \left(\Gamma\left(\frac{\beta+2}{\beta+1}\right)S\right)^{\beta+1}\right). \tag{26}$$

We shall refer to the level-spacing distribution implied by (25) as Berry–Robnik–Brody (BRB). It will be seen later that this phenomenological modelling is very successful leading to up to 100% statistical significance level.

In section 5 we show numerically, for a mapping on a two-dimensional torus, the existence of localized quantum states in a classically ergodic but strongly diffusive regime which gives rise to an excellent and convincing manifestation of the Brody distribution (see figure 8). Indeed this confirms the picture which has been outlined theoretically and qualitatively above.

4. Transition from near- to far-semiclassical regime in the billiard system

We can observe the theoretically described behaviour of the level-repulsion exponent β versus an effective value of $\hbar$ as predicted by equation (24) for a nearly integrable regime (small λ) of our billiard system. Since the billiards have a scaling property (all energy surfaces are dynamically equivalent) the semiclassical limit $\hbar \to 0$ is equivalent to the limit $E \to \infty$ with an effective value of Planck's constant $\hbar_{\mathrm{eff}} = E^{-1/2}$. So we have divided our spectrum for $\lambda = 0.1$ into several energy stretches and fitted the Brody distribution on each of them locally, for small spacings ($0 < S < 0.1$). Thus we have determined the level-repulsion exponent β as a function of energy E. This is an extremely difficult task since spectral stretches should be narrow enough in order to make the energy E well defined, but also wide enough in order to reduce the statistical fluctuations. We have also averaged $\beta(E)$ over three different, but similar, values of deformation parameter $\lambda = 0.095, 0.100$ and 0.105, but even then we can draw only qualitative conclusion (see figure 4) that $\beta(E)$ agrees with the asymptotic formula (24) of the previous section:

$$\beta(E) \propto E^{-1/2} \qquad E \to \infty. \tag{27}$$

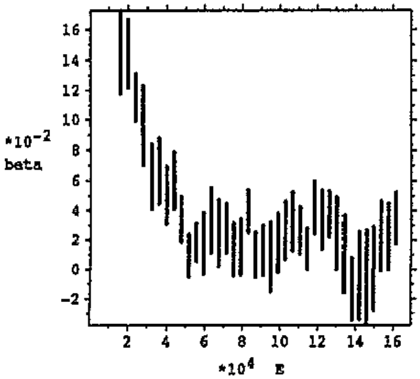

Figure 4. Here we show how the level-repulsion exponent β (determined by the Brody fit on the small spacing interval $0 < S < 0.1$) behaves as a function of the energy E for the nearly-integrable (KAM) regime of our billiard system (near $\lambda = 0.1$). To determine β as locally (with respect to the energy E) as possible we have used energy stretches containing 4000 levels. The positions of the bars denote the averaged β over three different nearby values of the deformation parameter $\lambda = 0.095, 0.100, 0.105$ and two parities whereas the heights of the bars are the estimates of the $\pm 1\sigma$ statistical error. Qualitatively we can confirm the dependence $\beta(E)$ of (24) but for a detailed quantitative analysis the numerics are still much too crude.

5. The standard map on a torus: going from Brody-like to the Berry–Robnik regime

Now let us define another dynamical system, following Prosen and Robnik (1994a), whose phase space is just a compact two-dimensional torus $T_2 = \{(x, y); x, y \in [-\pi, \pi)\}$, where the periodic coordinates x and y will be called position and momentum, respectively. The system's dynamics will simply be given by consecutive applications of 'free motions' $U_{\rm free}(x, y) = (x + y, y)$ and 'kicks' $U_{\rm kick}(x, y) = (x, y - a\sin(x))$. The most useful is the symmetric representation of the evolution mapping U,

$$U = U_{\rm kick}^{1/2} \circ U_{\rm free} \circ U_{\rm kick}^{1/2} \tag{28}$$

where $U_{\rm kick}^{1/2}(x, y) = (x, y - a\sin(x)/2)$. Our mapping (28) is, in fact, the standard (Chirikov) map on a torus T_2 rather than on a cylinder and its representation is dynamically (canonically) equivalent to the usual representation $U_{\rm kick} \circ U_{\rm free}$. It possesses two symmetries, namely the time-reversal symmetry $T(x, y) = (x, -y)$, $T \circ U \circ T = U^{-1}$, and parity $P(x, y) = (-x, -y)$, $P \circ U \circ P = U$.

Since the classical phase space is compact, the quantum Hilbert space is finite-dimensional and its dimension n determines the dimensionless value of the effective Planck's constant $\hbar_{\rm eff} = 2\pi/n$. Let n be an even number $n = 2m$. The position and momentum eigenstates denoted by $|x_k\rangle$ and $|y_l\rangle$ can be defined through the relation $\langle x_k|y_l\rangle = n^{-1/2}\exp((in/2\pi)x_k y_l)$, where our choice $x_k = (2\pi/n)(k - \frac{1}{2})$, $y_l = (2\pi/n)(l-1)$, $k, l = 1 \ldots n$, warrants the single-valuedness on the torus T_2. The quantization procedure is now almost obvious: the quantum unitary evolution propagator $\widehat{U}$ is decomposed to products of free motions $\widehat{U}_{\rm free}$ and kicks $\widehat{U}_{\rm kick}$ in precisely the same way as the classical one (28) where quantum analogues for the kick and the free motion are diagonal in position and momentum representation, respectively,

$$\widehat{U}_{\rm kick} = \sum_k \exp\left(\frac{in}{2\pi}a\cos(x_k)\right)|x_k\rangle\langle x_k| \qquad \widehat{U}_{\rm free} = \sum_l \exp\left(-\frac{in}{2\pi}\frac{y_l^2}{2}\right)|y_l\rangle\langle y_l|. \tag{29}$$

The phases of the diagonal elements in (29) are (when divided by $n/2\pi$) just the classical generating functions which generate the classical mapping (28). Therefore as $n \to \infty$ the quantum evolution approaches the classical dynamics. There exists a simple closed-form expression for the propagator in position representation

$$\langle x_k|\widehat{U}|x_{k'}\rangle = \frac{1}{\sqrt{n}}\exp\left[\frac{in}{2\pi}\left(\tfrac{1}{2}(x_k - x_{k'})^2 + \tfrac{1}{2}a\cos(x_k) + \tfrac{1}{2}a\cos(x_{k'})\right)\right] \tag{30}$$

which is the discrete time analogue of the well known infinitesimal propagator $\exp[(i/\hbar)((x - x')^2/2m\,dt - (V(x) + V(x'))\,dt/2)]$ for the general continuous Hamiltonian case. Using the symmetry under parity P one can further reduce the n-dimensional unitary matrix $U_{kk'} = \langle x_k|\widehat{U}|x_{k'}\rangle$ to two ($m = n/2$)-dimensional unitary matrices $U_{kk'}^{\sigma} = \langle x_k\sigma|\widehat{U}|x_{k'}\sigma\rangle$ where $|x_k\sigma\rangle$ are parity preserving position eigenstates $|x_k\sigma\rangle = 2^{-1/2}(|x_k\rangle + \sigma|-x_k\rangle)$, $k = 1 \ldots m$ and $\sigma = \pm 1$ is a parity eigenvalue. Of course, quantization can also be worked out for odd values of n but it is physically less transparent so we have used only even values of n in our numerical example. The same system but in a slightly different formulation, has been introduced and studied by Izrailev (1986, 1990).

We have diagonalized symmetric (due to time reversal) and unitary matrices U_{kl}^{σ} as far in the semiclassical limit $m \to \infty$ as possible. Spectra for both parities and several consecutive values of m were joined together in order to obtain statistically significant results. Clearly, due to the time-reversal symmetry the GOE (or, strictly speaking COE) statistics will apply to irregular level sequences. This is a high-quality resolution test of the

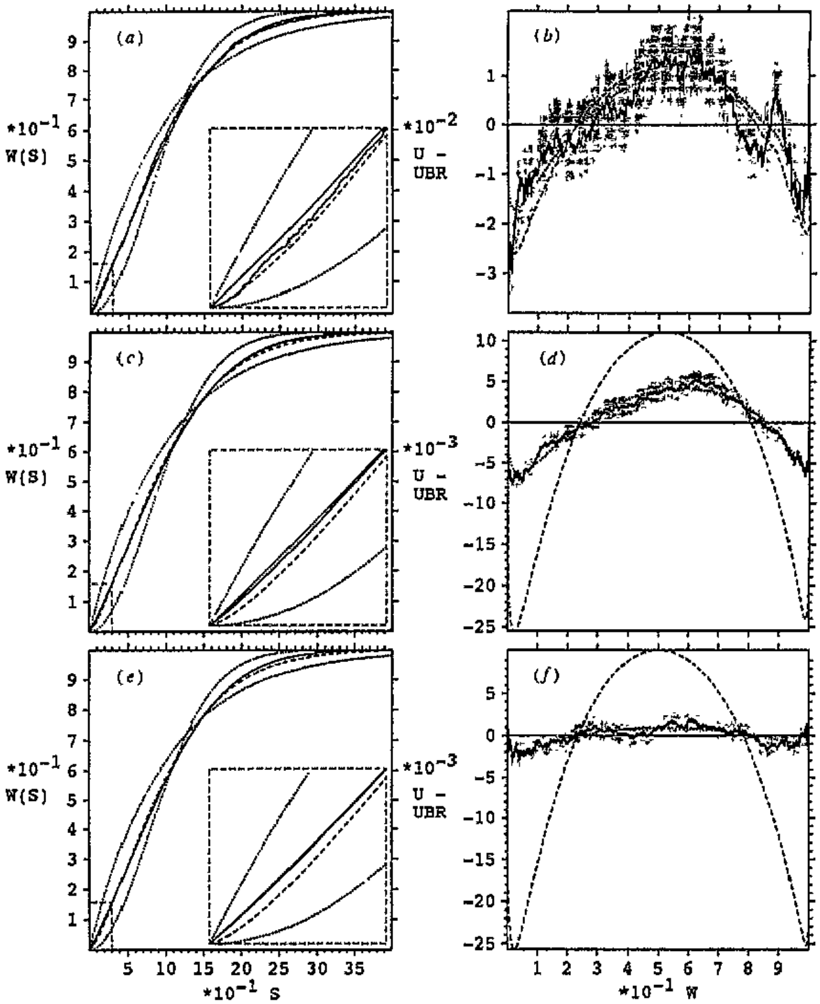

Figure 5. The figure illustrates the transition from the Brody-like to Berry–Robnik regime for the compact standard mapping system (30) at $a = 1.8$, at three stretches of m: (a), (b) 11 . . . 40 (1220 spacings), (c), (d) 301 . . . 400 (70 100 spacings), and (e), (f) 3991 . . . 4000 (79910 spacings) and both parities. The cumulative level-spacing distribution $W(S)$ ((a), (c), (e)) and the deviation of the U-function from the best-fit Berry–Robnik curve $U(W(S)) - U(W_2^{BR(\infty)}(S, \rho_1))$ versus $W(S)$ ((b), (d), (f)) is shown. On the left-hand side ((a), (c), (e)) the dotted curves are the limiting Poisson and GOE cases, the broken curve is the best-fit Brody distribution, the thin full curve is the best-fit Berry–Robnik curve and the thickest full curve is the numerical one. Small spacing regions $0 < S < 0.3$ are shown in magnified windows. On the right-hand side ((b), (d), (f)) the Berry–Robnik curve is just the abscissa, Brody is broken, while the dotted curve is the best-fit BRB distribution which excellently matches the data at all m, and the grey band indicates the $\pm 1\sigma$ statistical uncertainty of the numerical data.

Berry–Robnik formula so we have investigated the cumulative level-spacing distribution $W(S) = \int_0^S d\sigma\, P(\sigma)$ rather than the probability distribution $P(S)$ itself, since the latter suffers from arbitrariness of binning. We have applied a least-squares fit of the two-component Berry–Robnik formula with estimated one-sigma uncertainties for the numerical data $\delta W = \sqrt{W(1 - W)/N}$, where N is the total number of numerical spacings, (see PR),

Table 2. This table illustrates the transition (as average m increases) from near- to far-semiclassical regime (by using quasi-energy levels for each m from the interval $m_1 \leqslant m \leqslant m_2$ (columns 1 and 2)) for the two mixed regimes of the compact standard map $a = 1.3$ and $a = 1.8$. We show all the relevant parameters which were obtained by the least-squares fit of the level-spacing distribution with the three theoretical distributions: Berry–Robnik (columns 3 and 4), Brody (columns 5 and 6), and combined BRB (columns 7–9). The total number of spacings (for both parities) is equal to $N = m_2(m_2+1) - m_1(m_1 - 1)$. One can see that the transition is faster at $a = 1.8$ than at $a = 1.3$ (this is probably due to the less diffusive classical dynamics inside the chaotic component) whereas combined BRB distribution works excellently at all values of m.

m_1	m_2	β^{B}	χ^2/N	ρ_1^{BR}	χ^2/N	β^{BRB}	ρ_1^{BRB}	χ^2/N
$a = 1.3$, $\rho_1^{\mathrm{cl}} = 0.372$								
200	300	0.146	6	0.531	13	0.384	0.360	0.5
425	575	0.170	14	0.500	59	0.356	0.292	0.2
950	1050	0.188	27	0.476	77	0.401	0.292	0.2
1981	2020	0.207	40	0.449	48	0.488	0.311	0.4
3991	4000	0.237	44	0.409	13	0.639	0.330	0.8
7991	8000	0.240	126	0.400	6	0.789	0.365	0.6
$a = 1.8$, $\rho_1^{\mathrm{cl}} = 0.265$								
200	300	0.356	36	0.302	6	0.748	0.246	0.3
425	575	0.370	118	0.291	14	0.769	0.241	0.2
950	1050	0.380	209	0.282	6	0.843	0.253	0.2
1981	2020	0.386	167	0.277	5	0.843	0.247	0.2
3991	4000	0.392	95	0.272	0.9	0.891	0.255	0.2
7991	8000	0.388	220	0.273	0.3	0.927	0.265	0.5

and evaluated the χ^2 test. Moreover, we had to use the true infinite-dimensional GOE statistics to model chaotic spectral subsequence instead of the commonly used Wigner surmise, since, as deep in the semiclassical limit as we were able to go to (at $m = 8000$), we could clearly detect considerable differences. On the other hand we have also compared our data with the phenomenological Brody model of power-law level repulsion (8), which was reported by many authors (Wintgen and Friedrich 1987, Hönig and Wintgen 1989, PR, Ganesan and Lakshmanan 1994) to provide a statistically significant fit to physical data at practically accessible energies (i.e. effective values of $\hbar$). For a more refined analysis we have also used the U-representation of the level-spacing distribution (11). We have plotted $U(W(S)) - U(W_2^{\mathrm{BR}(\infty)}(S, \rho_1))$, where $W_2^{\mathrm{BR}(\infty)}$ is the best-fit two-component Berry–Robnik cumulative level-spacing distribution (based on infinite-dimensional GOE statistics), versus $W(S)$ (see figures 5 and 6). Here the density of equally weighted numerical points is constant along the abscissa so that the information really is uniformly distributed over the graph.

We have observed a very slow convergence towards the semiclassical limit, which is characterized by a smooth transition from the power-law Brody-like regime in the near-semiclassical limit towards the ultimate Berry–Robnik regime in the far semiclassical limit, as illustrated in figure 5 for the system (30) at $a = 1.8$. The quasi-universal Brody-like regime (with the fractional power-law level repulsion) has been clearly and statistically significantly demonstrated in (PR), and in section 2 of this paper. The origin of this phenomenon has been explained and understood theoretically in section 3. Thus the transition exemplified in figure 1 is very typical.

But for $a = 1.8$ and $m = 8000$ (numerical data were collected for $m = 7991 \ldots 8000$ and both parities) we have clearly reached the Berry–Robnik regime of the far-semiclassical limit which is reflected in the fact that the matching between numerics and the best-fit Berry–Robnik curve becomes really excellent (100% confidence level) (see figure 6). The value

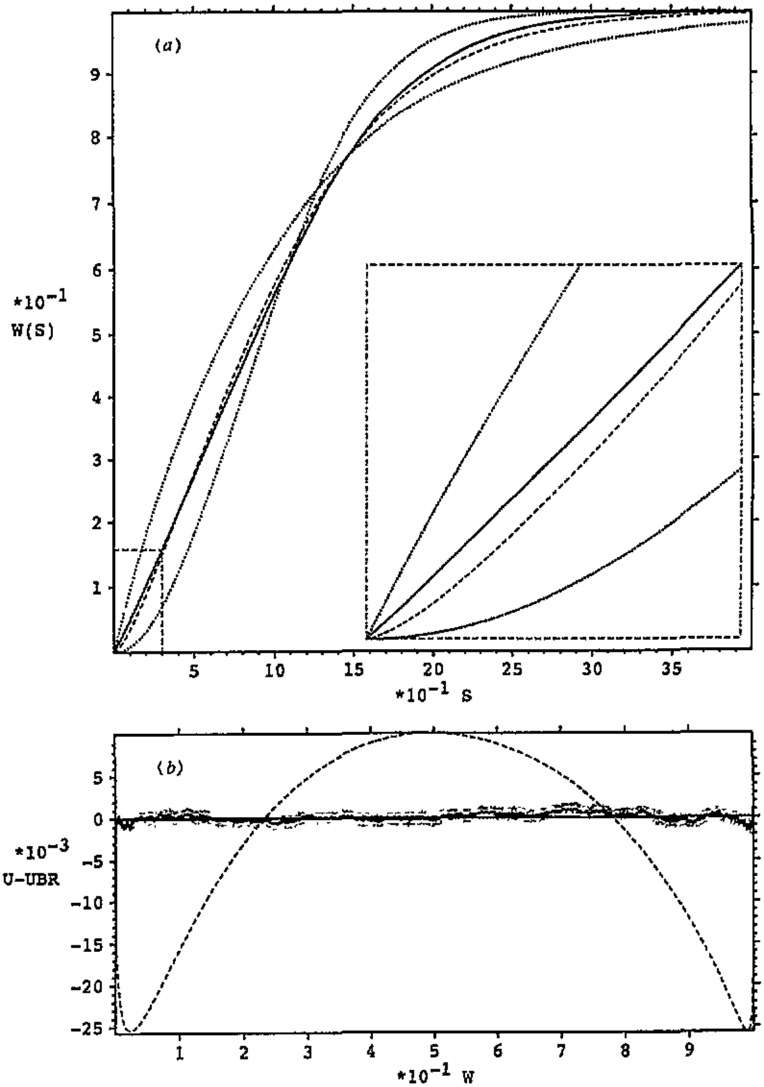

Figure 6. (a) Cumulative level-spacing distribution and (b) its U-function for the highest lying range of $m = 7991\ldots8000$ (159 910 spacings) in the standard mapping system on a torus. The meaning of the curves is the same as in figure 5 and also $a = 1.8$. In (a) the standard representation the theoretical and numerical curves are completely overlapping, and the agreement between the best-fit Berry–Robnik curve with $\rho_1 = 0.273$ and numerics is really excellent since the value of $\chi^2 = 45\,000$ is 3.5 times smaller than the number of spacings.

of the parameter $\rho_1 = 0.2727(1 \pm 0.9\%)$ only deviates by $\approx 3\%$ from the classical regular volume $\rho_1^{\rm cl} = 0.265(1 \pm 0.8\%)$. For larger values of ρ_1 closer to integrability (smaller values of parameter a) the convergence is even slower, e.g. for $a = 1.3$ ($\rho_1 = 0.372$) there are still tiny but detectable deviations between the theory and numerics even at $m = 8000$. As the value of m decreases the discrepancy between the quantal and classical value of ρ_1 increases, where the former is being typically larger than the latter. On the other hand in the whole range $1 < m < 8000$ (from near to far semiclassics) the two parameter (ρ_1, β) BRB model has turned out to be highly satisfactory (100% confidence level). See table 2 where

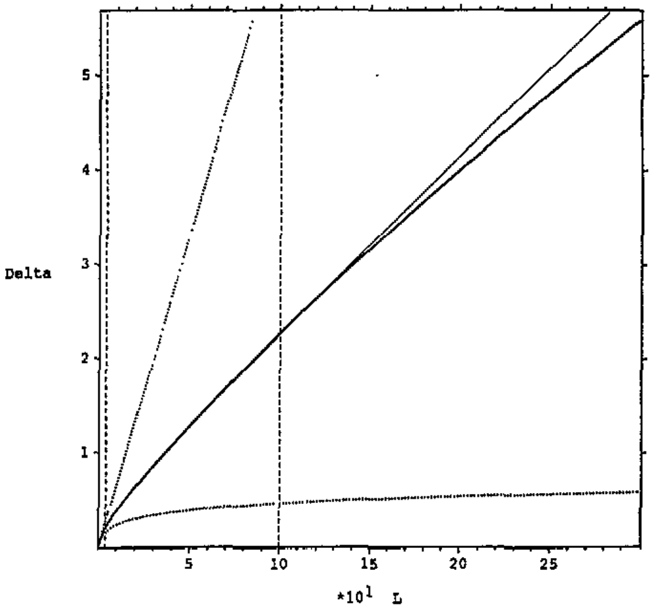

Figure 7. The delta statistics $\Delta(L)$ is shown at $a = 1.8$ for the highest lying range of $m = 7991\ldots8000$ in the quantum standard map on the torus. The dotted curves are the limiting Poisson and GOE curves, the thin full curve is the best-fit Seligman–Verbaarschot formula (5) with $\rho_1 = 0.274$, and the thick full curve shows the numerical data. The vertical broken lines indicate the region where the least-squares fit is applied. The theory starts to deviate from numerics above $L \approx 150$ where the saturation effects set in.

we show the relevant parameters of all the three distributions after the best-fit procedure. We have also found a significant fit to the semiclassical ansatz for the delta statistics $\Delta(L)$ (5) (at $m = 8000$) with the best-fit value of the parameter $\rho_1 = 0.274(1 \pm 1.5\%)$ deviating by 3.3% from the classical value (figure 7). The fit was on the interval $0 \leqslant L \leqslant 100$ which is—as judged *a posteriori*—safely below the saturation region (Berry 1985).

So far we have been talking about the Brody-like regime in a KAM-like transition from integrability to chaos for which, in section 3, we have given a theoretical explanation. Now we will discuss a dynamical system which is ergodic but strongly diffusive (slow classical diffusion implying an effective classical localization) leading to the quantum localization of eigenstates and Brody-like spectral statistics. To this end we have slightly generalized the standard mapping introduced above by defining $U_{\text{kick}}(x, y) = (x, y - a\sin(x) + b\sin(59x))$, where a and b are small, specifically $a = b = 0.1$. The classical mapping is then fully specified by (28) and correspondingly its quantal analogue by (30) where the two terms $\frac{1}{2}a\cos(x)$ must be supplemented by adding the two terms $-\frac{1}{2}(b/59)\cos(59x)$, respectively. The classical dynamics has been carefully inspected and indeed found to be diffusive ergodic. In figure 8 we show the result of the spectral statistical analysis. The Brody distribution is seen to be a good fit to the numerical data. A smooth transition from almost Poisson towards GOE is clearly demonstrated. Technically, in order to verify the consistency we have also fitted the data by the BRB distribution with the result that ρ_1 is close to zero (in fact slightly negative) and β close to the Brody value.

The presented numerical material and the theoretical considerations provide firm support for our opinion that the two-parameter BRB distribution represents the most general family of

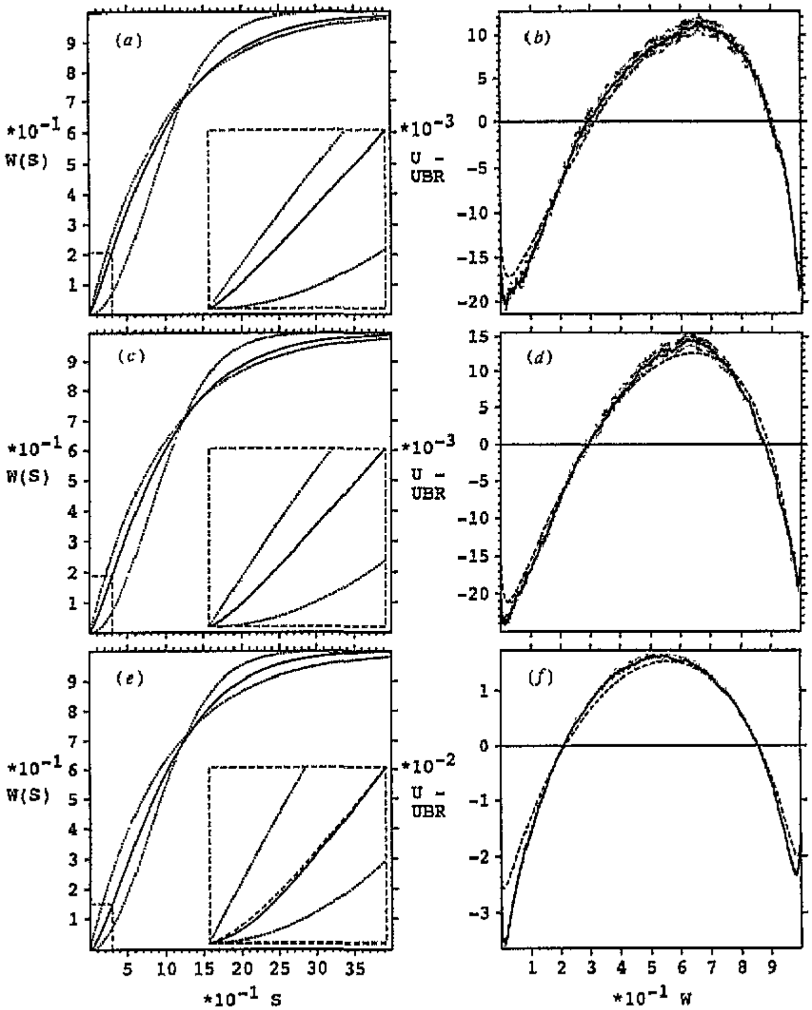

Figure 8. The cumulative level-spacing distribution $W(S)$ ((*a*), (*c*), (*e*)) and the deviation of the U-function from the best fitting Berry–Robnik curve $U(W(S))-U(W_2^{\mathrm{BR}(\infty)}(S,\rho_1))$ versus $W(S)$ ((*b*), (*d*), (*f*)) is shown for the ergodic-diffusive modified standard mapping (see section 5) and for three different stretches of $m=\pi/\hbar_{\mathrm{eff}}$: (*a*), (*b*) $m=451\ldots550$, (*c*), (*d*) $m=976\ldots1025$, (*e*), (*f*) $m=1981\ldots2020$. The graphical representation is analogous to the figure 5 but here we do not plot the best-fit Berry–Robnik and BRB distributions (for the obvious reasons) but we only compare our data with the Brody distribution (broken curves) which turns out to be surprisingly good fit especially in the near-semiclassical regime (smaller m). The corresponding values of the level-repulsion parameter are (*a*), (*b*) $\beta=0.16$, (*c*), (*d*) 0.23, (*e*), (*f*) 0.39.

distributions capturing the double transition: from integrability to chaos and from near to far semiclassics (from Brody-like to Berry–Robnik) in the case where we have only one large dominant classically chaotic region, or alternatively from strong localization to extendedness in a diffusive ergodic regime. In this model the localization effects on chaotic components are captured by the parameter β and as a consequence of that the (best-fit) quantal ρ_1^{BRB} is strictly smaller than the classical one ρ_1^{cl} which, in turn, is a consequence of the fact that the exponential tails of the quantal chaotic states penetrate into the classically regular regions.

This circumstance is illustrated numerically in table 1 for the billiard system and in table 2 for the mapping system. The discrepancy between classical and quantal values of ρ_1 is larger in the billiard system than in the mapping system because the effective $\hbar$ is larger in the former system.

6. The structure of eigenstates and their relevance for the semiclassical behaviour of level statistics

It appears that the geometry of eigenstates (configurational wavefunctions and their Wigner distributions in the quantum phase space) and their morphological types correlate with the statistical properties of the energy spectra (and of other observables). We have systematically surveyed a vast amount of representative eigenstates both in the billiard system and in the compactified standard map in various regimes between integrability and irregularity covering also the transitional regime from Brody-like to Berry–Robnik behaviour. The old idea by Percival (1973) of classifying the energy levels and the corresponding eigenstates into *regular* and *irregular* is being confirmed in a rather convincing manner. (The details of this research will be published in a separate paper (Prosen and Robnik 1994b), but here we expound some general conclusions.) All states can be classified clearly either as regular or irregular in the sense that they 'live' exclusively inside a classically regular or irregular region in the phase space, respectively, except for those cases where the two states are almost degenerate (exhibiting a narrowly avoided crossing), and allow for a mixing of the two classes of states (*mixed states*). But the measure of mixed states vanishes in the semiclassical limit hence the validity of the Percival conjecture.

However, the majority of irregular states for systems in the transition region of mixed type classical dynamics in the near-semiclassical limit is localized or even strongly localized in the sense that they effectively occupy only a strict subset of the underlying classically chaotic region. But as $\hbar \to 0$ in the far-semiclassical limit we indeed observe a general tendency towards uniform extendedness: the so-called principle of uniform semiclassical condensation applies (see, for example, Li and Robnik 1994a, Berry 1977, Robnik 1988). The locally averaged Wigner functions tend to the classical probability density on the given classical invariant component. Thus in the ultimate semiclassical limit we see that the quantum eigenstates are condensed on disjoint regions in the phase space which implies little interaction between them giving rise to the statistical independence of the corresponding energy level sequences. This is the basis of the Berry–Robnik approach.

However, before we reach this far-semiclassical limit either the effects of tunnelling or the effects of localization of eigenstates give rise to the fractional power-law level repulsion and the Brody-like behaviour, as described in section 3.

The picture outlined in this section has also been revealed numerically in a study of the same billiard and of other billiards such as the Bunimovich stadium using a different numerical technique (Li and Robnik 1994a, b) which allows us to go higher into the semiclassical limit but has the disadvantage of systematically skipping pairs of almost degenerate states.

7. Discussion and conclusions

This paper reports on the first successful verification of the Berry–Robnik level-spacing distribution, which eliminates any doubts about the validity of the Berry–Robnik regime in the dispute concerning the ultimate semiclassical spectral statistics (in generic Hamiltonian systems), thereby supporting the view that the Berry–Robnik picture is the asymptotically exact description of level statistics. With present day supercomputer capabilities such a statistically significant analysis is possible only for one-dimensional time-dependent systems, like our compactified quantum standard map, because the so-called far-semiclassical limit is being formed very slowly as $\hbar \to 0$ and the effective $\hbar$ is related to the dimension n of matrices which need to be diagonalized via $\hbar \propto n^{-1/f}$ where f is the number of freedoms. Correspondingly, we have investigated the structure of eigenstates in phase space (Wigner and Husimi distributions) and we have also found very slow convergence to the ultimate uniform localization on classical invariant components in the semiclassical limit (Prosen and Robnik 1994b). Non-uniform localization on the chaotic region survives much higher in n than one would expect if it were just a consequence of slow classical transport in phase space due to partial barriers (Bohigas *et al* 1993).

In practice, however, it is very hard to reach the far-semiclassical limit and we typically observe Brody-like behaviour instead. This is characterized by the fractional power-law level repulsion at small S as explained in section 3: the origin of this phenomenon is either the validity of the tunnelling action formula (16) in cases of soft KAM chaos (which is theoretically completely understood), or some equivalent of that in cases of localization in classically diffusive ergodic regions (which is not yet understood quantitatively). In this paper we have offered massive numerical evidence for this type of behaviour.

Having understood that we propose a two-parameter family of level-spacing distributions which we call Berry–Robnik–Brody (BRB), since it is based on two-component Berry–Robnik geometry but the irregular level sequence is modelled by a Brody distribution rather than GOE, thereby taking into account the localization phenomena on the classically chaotic regions. This rather phenomenological formula has been tested very carefully and it turns out that it is 100% statistically significant in almost all cases.

In concluding we point out that the most important and outstanding open problem related to the research of the present paper is the precise and quantitative understanding and modelling of the localization phenomena on classically chaotic regions. The solution to this problem will lead us to the correct mathematical treatment of the Brody-like behaviour in classically diffusive ergodic regimes, as outlined in section 3.

As for the KAM soft chaos scenario explained and understood in section 3 we should mention a cluster of related ideas and problems embodied in our sparsed banded random-matrix ensembles (SBRME) published in Prosen and Robnik (1993b), which are intended to model KAM perturbed Hamiltonians in the basis of its integrable part. Here we again face some interesting mathematical problems related to localization phenomena. In our numerical analysis of SBRME so far we have confirmed the existence of the fractional power-law level repulsion and the relevance of the Brody-like behaviour.

Acknowledgments

One of us (MR) thanks Boris V Chirikov for many stimulating discussions, especially for pointing out the importance and relevance of localization for Brody-like behaviour and for suggesting a numerical study of these aspects in a classically diffusive ergodic system. Also, stimulating discussions with Oriol Bohigas, Giulio Casati, Fritz Haake and Hans A Weidenmüller are gratefully acknowledged.

The financial support by the Ministry of Science and Technology of the Republic of Slovenia is gratefully acknowledged.

References

Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
Berry M V 1977 *J. Phys. A: Math. Gen.* **10** 2083
——1985 *Proc. R. Soc.* A **400** 229
Berry M V and Robnik M 1984 *J. Phys. A: Math. Gen.* **17** 2413
Bohigas O 1991 *Chaos and Quantum Physics* ed M-J Giannoni, A Voros and J Zinn-Justin (Amsterdam: North-Holland) pp 87–199
Bohigas O and Giannoni M-J 1984 *Lecture Notes in Physics 209* (Berlin: Springer) p 1
Bohigas O, Tomsovic S and Ullmo D 1993 *Phys. Rep.* **223** 4
Brody T A 1973 *Lett. Nuovo Cimento* **7** 482
Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S M 1981 *Rev. Mod. Phys.* **53** 385
Bruus H and Stone A D 1994 *Preprint* Department of Physics, Yale University
Casati G and Chirikov B V 1994 *Quantum Chaos: Between Order and Disorder* ed G Casati and B V Chirikov (Cambridge: Cambridge University Press)
Chirikov B V 1991 *Chaos and Quantum Physics* ed M-J Giannoni, A Voros and J Zinn-Justin (Amsterdam: North-Holland) pp 443–545
Eckhardt B 1988 *Phys. Rep.* **163** 205
Frisk H 1990 *Preprint* NORDITA–90/46 S
Ganesan K and Lakshmanan M 1994 *J. Phys. B: At. Mol. Opt. Phys.* **27** at press
Giannoni M-J, Voros A and Zinn-Justin J 1991 *Chaos and Quantum Physics* (Amsterdam: North-Holland)
Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (New York: Springer)
Haake F 1991 *Quantum Signatures of Chaos* (Berlin: Springer)
Hönig A and Wintgen D 1989 *Phys. Rev.* A **39** 5642
Izrailev F M 1986 *Phys. Rev. Lett.* **56** 541
——1989 *J. Phys. A: Math. Gen.* **22** 865
——1990 *Phys. Rep.* **196** 299
Li Baowen and Robnik M 1994a *J. Phys. A: Math. Gen.* **27** 5509
——1994b *Preprint* CAMTP/94-8 (submitted to *J. Phys. A: Math. Gen.*)
——1994c *J. Phys. A: Math. Gen.* to be submitted
Markarian R 1993 *Nonlinearity* **6** 819
Mather J N 1982 *Ergodic Theory Dynamic. Sys.* **2** 3
Mehta M L 1991 *Random Matrices* (San Diego, CA: Academic)
Meyer H D 1986 *J. Chem. Phys.* **84** 3147
Meyer H D, Haller E, Köppel H and Cederbaum L S 1984 *J. Phys. A: Math. Gen.* **17** L831
Percival I C 1973 *J. Phys. B: At. Mol. Phys.* **6** L229
Prosen T and Robnik M 1993a *J. Phys. A: Math. Gen.* **26** 2371
——1993b *J. Phys. A: Math. Gen.* **26** 1105
——1994a *J. Phys. A: Math. Gen.* **27** L459
——1994b *J. Phys. A: Math. Gen.* to be submitted
Robnik M 1983 *J. Phys. A: Math. Gen.* **16** 3971
——1984 *J. Phys. A: Math. Gen.* **17** 1049
——1988 *Atomic Spectra and Collisions in External Fields* ed K T Taylor, M H Nayfeh and C W Clark (New York: Plenum) pp 265–74
——1994 *J. Phys. Soc. Japan Suppl.* A **63** 131
Seligman T H and Verbaarschot J J M 1985 *J. Phys. A: Math. Gen.* **18** 2227
Seligman T H, Verbaarschot J J M and Zirnbauer M R 1984 *Phys. Rev. Lett.* **53** 215
Stone A D and Bruus H 1993 *Physica* **189B** 43
——1994 *Surf. Sci.* at press
Wintgen D and Friedrich H 1987 *Phys. Rev.* A **35** 1464