

Transport in a one-dimensional isotropic Heisenberg model at high temperature

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Abstract. Magnetization transport in a one-dimensional isotropic spin 1/2 Heisenberg model is studied. It is shown that in a nonequilibrium steady state at high temperature and constant small driving the magnetization current depends on the system length L as $\sim 1/L^{0.5}$, meaning that the diffusion constant diverges as $\sim L^{0.5}$. Spectral properties of a superoperator governing the relaxation towards a nonequilibrium steady state are also discussed.

Keywords: spin chains, ladders and planes (theory), quantum transport in one dimension, quantum transport

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1. Introduction

The Heisenberg model of nearest-neighbor coupled spins is of high interest in theoretical as well as in experimental physics. The simplest variant is a one-dimensional (1D) chain of coupled spin-1/2 particles, described by the Hamiltonian

$$H = \sum_{j=1}^{L-1} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z, \tag{1}$$

in terms of Pauli matrices $\sigma_j^{x,y,z}$ at lattice site j . Using Jordan–Wigner transformation [1] it can be mapped to a system of spinless fermions whose Hamiltonian has a kinetic (hopping) term and a density–density interaction term, for fermions at neighboring sites. It therefore represents one of the simplest systems of strongly interacting fermions. The model (1) is exactly solvable by the Bethe ansatz [2]. Despite its solvability, time-dependent properties, like a time-dependent current autocorrelation function that is via a linear response theory directly related to the transport coefficient, are at present beyond the capabilities of exact methods. The isotropic one-dimensional Heisenberg model (1) is experimentally realized in so-called spin-chain materials [3], for instance in many cuprates. As of yet unexplained in these materials is a very high thermal conductivity along the axis of Heisenberg chains [4], believed to be due to contributions from Heisenberg chains and strongly influenced by impurities.

In the present work we shall study magnetization transport in the isotropic Heisenberg chain; for an overview of references on a more general anisotropic Heisenberg model see the introductions in [5, 6]. The isotropic Heisenberg model has been studied in the past; however, no definite conclusion about magnetization transport has been reached so far. Most studies focused on the Drude weight, whose non-zero value indicates a non-diffusive transport, usually just called ballistic. Using the Bethe ansatz, a non-zero Drude weight at all temperatures is advocated in [7], with $\sim 1/T$ behavior at high temperatures, while zero Drude weight at all temperatures is predicted in [8]. Exact diagonalizations [9], as well as conformal field theory [10] and quantum Monte Carlo approaches [11], also result in a finite Drude weight at high temperatures; see also [12]–[14]. Using a current

autocorrelation function obtained by exact diagonalization is non-conclusive [15] because long time scales exist. Extrapolation to the thermodynamical limit is very difficult with these results because exact diagonalization is limited to small systems, while quantum Monte Carlo, Bethe ansatz and conformal approaches have their own problems [6]. Low-energy bosonization calculation, together with the analysis of the current autocorrelation function from the density matrix renormalization group method, on the other hand, indicates [6] the presence of a diffusive contribution, also seen in quantum Monte Carlo calculations [16]. All of these results, some showing zero Drude weight, others showing non-zero Drude weight, or even the presence of a diffusive component, call for a more precise characterization of transport. Quantification just in terms of zero or non-zero Drude weight fails to distinguish between different variants of non-diffusive behavior.

A more detailed classification can be made for instance by studying how the current j scales with system length L if one fixes the difference of potentials at boundaries. Two extreme examples are the scaling $j \sim 1/L$ if a system obeys the Fourier law, and $j \sim L^0$ in the case of ballistic transport. However, as is well known from studies of classical transport [17], in many systems the scaling exponent is a real number, $j \sim L^{-\alpha}$, with $0 \leq \alpha \leq \infty$. Transport is called anomalous if the scaling exponent differs from 1, $\alpha \neq 1$. The exponent α is via the Fourier-like law that relates a transported quantity z and its current j , $j = -D \nabla z$, directly related to the diffusion constant D , resulting in the scaling $D \sim L^{1-\alpha}$. Under certain assumptions a heuristic argument with classical non-interacting particles leads to a connection between α and the exponent β of the spreading of disturbances as quantified by the variance σ^2 , $\sigma^2 \sim t^\beta$. The relation is $\beta = 2/(1 + \alpha)$ [18]. The regime of $1 < \beta \leq 2$, corresponding to $0 \leq \alpha < 1$, is called super-diffusion, while that of $0 \leq \beta < 1$, corresponding to $\alpha > 1$, is called sub-diffusion. Note that sub-ballistic transport with $\beta < 2$ means that the Drude weight is zero.

Recently, numerical simulations have shown [19] that $\alpha = 0.5$ in the isotropic Heisenberg model at an infinite temperature in the linear response regime. In the present work we shall extend these results by calculating the diffusion constant also at finite temperatures, showing that the scaling stays the same at high temperatures (higher than the exchange interaction). We shall also provide some other properties of the isotropic Heisenberg model, like the rate of relaxation to a nonequilibrium steady state.

2. The method

In order to be able to study nonequilibrium stationary states (NESS) we couple the system to reservoirs at the left and right chain ends. The two reservoirs are kept at different potentials, inducing a non-zero magnetization current through the chain. We describe reservoirs in an effective way using the Lindblad equation [20] for the density matrix ρ of the system,

$$d\rho/dt = i[\rho, H] + \mathcal{L}^{\text{dis}}(\rho) = \mathcal{L}(\rho), \quad (2)$$

where the dissipative linear operator \mathcal{L}^{dis} describing the bath is expressed in terms of Lindblad operators L_k , $\mathcal{L}^{\text{dis}}(\rho) = \sum_k ([L_k \rho, L_k^\dagger] + [L_k, \rho L_k^\dagger])$.

We use two kinds of reservoirs. To obtain NESS at infinite temperature, i.e., zero energy density, we use the so-called one-spin bath which is realized by two Lindblad operators at each end, $L_1^L = \sqrt{1 - \mu} \sigma_1^+$, $L_2^L = \sqrt{1 + \mu} \sigma_1^-$ at the left end and

$L_1^R = \sqrt{1+\mu}\sigma_n^+$, $L_2^R = \sqrt{1-\mu}\sigma_n^-$ at the right end, $\sigma^\pm = (\sigma^x \pm i\sigma^y)/2$, with $\sigma^{x,y,z}$ being Pauli matrices.

To simulate NESS at a finite temperature we use the so-called two-spin bath, in which one has 16 Lindblad operators acting on two boundary spins at each end. The form of these 16 operators is complicated and we do not state it explicitly. They are obtained by demanding that the stationary equation for two boundary spins, described by $\rho^{(2)}$, $\mathcal{L}^{\text{dis}}(\rho^{(2)}) = 0$, has for a solution a grand canonical state $\rho^{(2)}$. The targeted grand canonical state $\rho^{(2)}$ is obtained by calculating it from a small chain of eight spins, $\rho^{(2)} \sim \text{tr}_{3,\dots,8}(\exp(-H/T_{L,R} + \mu_{L,R}M))$ (tracing is performed over six spins in a chain of length 8), where $M = \sum_{j=1}^L \sigma_j^z$ is the total magnetization and $T_{L,R}$ and $\mu_{L,R}$ are the imposed temperature and potential at the left/right ends of the chain. Details of the implementation can be found in [21]. Note that $\rho^{(2)}$ is used only to generate appropriate Lindblad operators that will simulate finite temperature. Once a NESS is found for a large chain (of up to 256 spins), a real system's temperature will be determined by calculating expectation values of observables in the NESS. Our results do not depend on details of the Lindblad operators used in the simulation (and therefore also not on details of $\rho^{(2)}$ used in deriving them); the only goal is to impose a finite energy density in the NESS. The driving parameter $\mu_{L,R}$ (or μ) will always be small, as we are interested in the linear response regime. For stationary properties under maximally strong one-spin driving, see [22].

For both kinds of reservoirs the NESS, simply denoted by ρ in the rest of the paper, is found by evolving an arbitrary initial state $\rho(0)$ with the Lindblad equation for sufficiently long time, until a nonequilibrium stationary state $\rho = \lim_{t \rightarrow \infty} \rho(t)$ is reached. To calculate time evolution of $\rho(t)$ we use a time-dependent density matrix renormalization group (tDMRG) method, as described in [21].

In section 3.3 we will be interested also in spectral properties of the Liouville superoperator \mathcal{L} (2). Because a detailed knowledge of the eigenvalues of \mathcal{L} cannot be obtained by a simple implementation of the tDMRG method that we use, we have instead used an exact diagonalization on somewhat smaller systems.

3. Results

3.1. Determining the temperature

In all our simulations, using one-spin or two-spin baths, we use a weak driving, $\mu_L = -\mu = -0.02$ and $\mu_R = +\mu = 0.02$. Symmetric driving with respect to left/right ends imposes a NESS state with the average magnetization being zero. In a two-spin bath, with which we can simulate finite temperature states, the imposed temperatures are the same at the left end and the right end: $T_L = T_R = T_{\text{imp}}$. A consequence of this is that the energy density in the NESS,

$$h_j = \langle \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z \rangle, \quad (3)$$

is independent of the position index j and the energy current in the NESS is therefore zero. $\langle \rangle$ denotes the expectation value in the NESS, $\langle A \rangle = \text{tr} \rho A$. Because driving is weak, the NESS is locally close to equilibrium. We can therefore determine [23] the temperature of the NESS, called a 'measured' temperature T_{meas} , by equating the expectation value

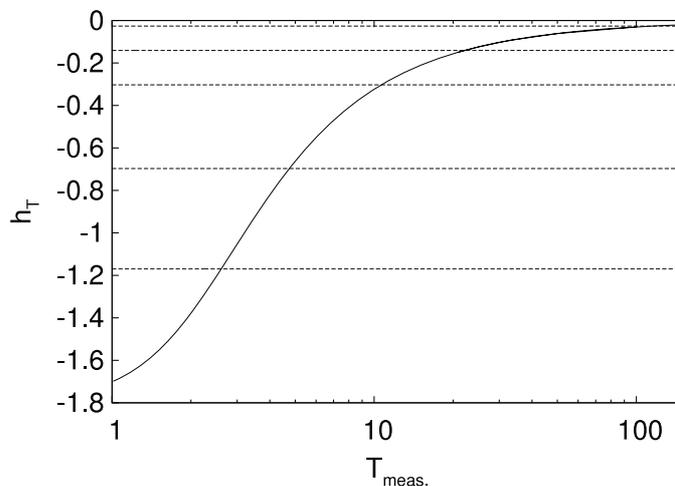


Figure 1. Average energy density in a canonical state (4) of the isotropic one-dimensional Heisenberg model (1). Horizontal lines show energy densities h_j in the NESS states used in the paper.

of the energy density to the one expected in a canonical state, $h_j = h_T$:

$$h_T = \frac{\text{tr}(\rho_T H)}{L-1}, \quad \rho_T = \frac{\exp(-H/T_{\text{meas}})}{\text{tr} \exp(-H/T_{\text{meas}})}, \quad (4)$$

and solving for T_{meas} . The NESS can be approximated by a canonical state because the average potential μ_{meas} is zero. We have checked that the NESS state is indeed close to the canonical one by verifying that the expectation values of nearest-neighbor observables in the bulk of the NESS agree with the canonical ones within $\mathcal{O}(\mu_{L,R})$. For instance, for the NESS with $T_{\text{meas}} \approx 4.8$ we have one-point expectation values $\langle \sigma_k^{x,y} \rangle < 10^{-5}$ (they are non-zero due to the tDMRG truncation error), $\langle \sigma_k^z \rangle \sim \mathcal{O}(\mu_{L,R})$, two-point expectation values $\langle \sigma_k^x \sigma_{k+1}^x \rangle \approx \langle \sigma_k^y \sigma_{k+1}^y \rangle \approx \langle \sigma_k^z \sigma_{k+1}^z \rangle \approx h_T/3$, $\langle j_k \rangle \sim \mathcal{O}(\mu_{L,R})$, while the other two-point nearest-neighbor expectation values are smaller than 10^{-5} . In the thermal canonical state all one-point expectation values are zero, while the two-point nearest-neighbor ones agree with the ones in the NESS. The three-point expectation values in the NESS are all smaller than 10^{-5} , except $\langle \sigma_k^x \sigma_{k+1}^x \sigma_{k+2}^z \rangle$ and $\langle \sigma_k^y \sigma_{k+1}^y \sigma_{k+2}^z \rangle$ (as well as those for all permutations of the three Pauli matrices occurring in these two operators) which are of order 10^{-4} . In the canonical state all three-point nearest-neighbor expectation values are zero. Some four-point nearest-neighbor expectation values in the NESS are non-zero and of size ~ 0.05 , for instance operators like $\sigma_k^x \sigma_{k+1}^x \sigma_{k+2}^x \sigma_{k+3}^x$, $\sigma_k^x \sigma_{k+1}^x \sigma_{k+2}^z \sigma_{k+3}^z$ or $\sigma_k^x \sigma_{k+1}^x \sigma_{k+2}^y \sigma_{k+3}^y$ (and permutations of these four operators). The corresponding canonical expectation values are also non-zero and within $\mathcal{O}(\mu_{L,R})$ of the NESS expectation values. All canonical expectation values mentioned have been calculated at high temperatures by an exact diagonalization of small systems, while the imaginary-time tDMRG method has been used at smaller temperatures.

In figure 1 we show the dependence of the canonical energy density on temperature. It turns out that the boundary effects with our two-spin bath (see later) get increasingly stronger with lowering of the imposed temperature. Because the operator entanglement

Table 1. Data for NESS states used in the paper. For $T_{\text{imp}} = \infty$ we use a one-spin bath, for others a two-spin bath. The measured temperature is determined by equating the energy density h_j in the NESS to the canonical expectation value h_T . The non-zero h_j for $T_{\text{imp}} = \infty$ is due to truncation errors of the tDMRG method.

T_{imp}	h_j	T_{meas}
∞	0.0004	∞
50	-0.026	120
10	-0.141	22
5	-0.30	10.6
2	-0.69	4.8
0.2	-1.17	2.6

of the NESS ρ also increases, simulations get increasingly more difficult. We are therefore not able to reach very low temperatures. The temperatures that we used are listed in table 1. Note that the minimal temperature achieved, $T_{\text{meas}} = 2.6$, would in the spin notation where $H = \sum_{j=1}^{L-1} s_j^x s_{j+1}^x + s_j^y s_{j+1}^y + s_j^z s_{j+1}^z$, with $s^{x,y,z} = \sigma^{x,y,z}/2$, correspond to $T_{\text{meas}} = 0.65$. The exchange interaction in SrCuO₂ is approximately $J/k_B \approx 2000$ K. Temperatures in the experiments [4], which are of the order ~ 100 K, therefore correspond to the dimensionless temperature $T_{\text{meas}} \approx 0.2$ in our Pauli notation. Such low temperatures are unfortunately not reachable with our reservoirs [23].

3.2. Magnetization profiles and the current

The main quantity that we consider is the scaling of the expectation value of the magnetization current in the NESS with the system size L . The magnetization current operator is

$$j_k = 2(\sigma_k^x \sigma_{k+1}^y - \sigma_k^y \sigma_{k+1}^x), \quad (5)$$

and we denote its expectation value¹ in the NESS, which is independent of the site index k , simply by $j = \langle j_k \rangle$. In figure 2 we show results at various temperatures, all for the same driving, $\mu_{L,R} = \pm 0.02$. In addition to NESS results obtained by the tDMRG approach, we also show the ones obtained by numerically exactly solving [24] the master equation (2). The reason that in general the current j decreases with decreasing temperature is also a consequence of increasing boundary resistances due to two-spin reservoirs used. There is a magnetization jump at the boundary, so the first and the last spin have magnetization smaller than the imposed $\mu = 0.02$. This can be seen in figure 3 for data at finite temperatures. The magnetization profiles at all temperatures from table 1 (except the ones at $T_{\text{meas}} = 2.6$) can be well described by the scaling function $\langle \sigma_r^z \rangle = k(2\mu/\pi) \arcsin(x)$, where $x = 2(r - 0.5)/L - 1$ is a scaled position and k is a temperature-dependent prefactor, effectively taking into account for boundary jumps. For instance, at $T_{\text{meas}} = 10.6$ (left frame in figure 3) it is $k \approx 0.65$. Because of the boundary

¹ If we had a prefactor 1/4 in the H , i.e., had spin variables instead of Pauli matrices, the temperature T , energy density $\langle h \rangle$ and diffusion constant D given in the paper would have to be divided by 4, whereas the magnetization current given in the paper would have to be divided by 8.

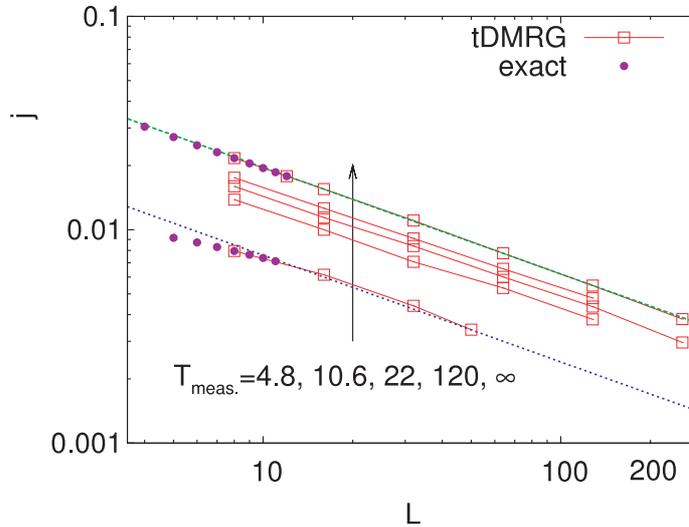


Figure 2. Scaling of current j with L for different temperatures T_{meas} , all at the same driving $\mu_{L,R}$. The circles were obtained using the numerically exact NESS state while the squares were obtained using the tDMRG. The data at $T_{\text{meas}} = \infty$ are the same as in [19], apart from the new data point for $L = 256$. The two straight lines overlapping with $T_{\text{meas}} = 4.8$ and ∞ data are $\sim 1/L^{0.5}$.

jumps, to properly account for the scaling of j with L , i.e., to assess the validity of the Fourier law,

$$j = -D \nabla_r \langle \sigma_r^z \rangle, \quad (6)$$

we have to scale the current with the actual magnetization difference given by $\langle \sigma_1^z \rangle - \langle \sigma_L^z \rangle \equiv \Delta z$. This is shown in figure 4. From the figure we can estimate that the current is

$$j = -1.55 \frac{\langle \sigma_1^z \rangle - \langle \sigma_L^z \rangle}{L^{0.5}}, \quad (7)$$

independent of the temperature (energy density), at least for $T_{\text{meas}} \gtrsim 5$. In this range of temperatures the diffusion constant D (6) therefore scales as

$$D \approx 1.55 L^{0.5}. \quad (8)$$

At lower temperatures it is difficult to assess whether the scaling is still the same. Data in figure 4 for $T_{\text{meas}} = 2.6$ and 4.8 show larger current than predicted by equation (7); however, one explanation could be that the length at which the asymptotic behavior (7) begins gets larger than the sizes studied. In the right frame of figure 3 we can for instance see that the asymptotic arcsin x magnetization profile is at $T_{\text{meas}} = 4.8$ not yet reached for $L = 32$, whereas at higher temperatures this happens already for smaller values of L (left frame).

Scaling of the current $j \sim 1/\sqrt{L}$ or, equivalently, of the diffusion constant $D \sim \sqrt{L}$ can be used to show in a non-rigorous way the spatial dependence of the magnetization. We shall use the Fourier law (6) with a space-dependent diffusion constant $D(r)$ at site r .

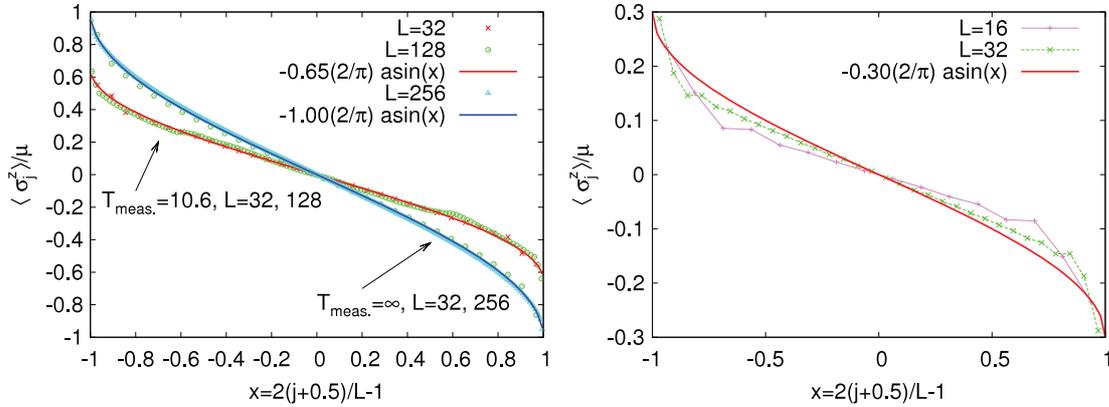


Figure 3. Left: magnetization profiles at $T_{\text{meas}} = \infty$ and at $T_{\text{meas}} = 10.6$. At lower temperature the magnetization exhibits jumps at the boundary. At all temperatures the profile is well described by the scaling function $\arcsin(x)$. Right: at lower $T_{\text{meas}} = 4.8$ the convergence of profiles to $\arcsin(x)$ seems to happen at larger sizes L than at higher temperatures.

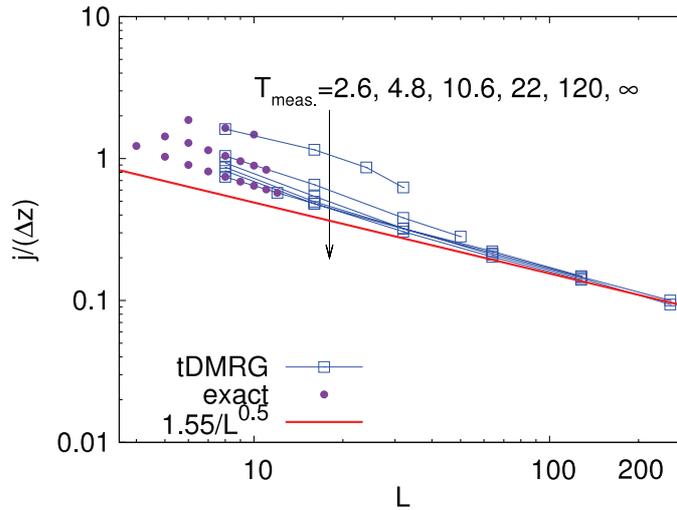


Figure 4. Current divided by $\Delta z = \langle \sigma_1^z - \sigma_L^z \rangle$ for differing L and temperature. For sufficiently large L they all seem to converge to the same line $\sim 1.55/L^{0.5}$.

Because equation (8) tells us that the diffusion constant gets larger for larger chains, close to boundaries, the local diffusion constant should become smaller. Assuming a square-root scaling we must have $D(r) \propto \sqrt{r(L-r)/L}$. This gives a differential equation

$$j = -\frac{\text{const.}}{\sqrt{L}} = -\sqrt{\frac{r(L-r)}{L}} \frac{dz}{dr}, \quad (9)$$

where $z = \langle \sigma_r^z \rangle$. Integrating the above equation with appropriate boundary conditions one immediately gets the profile $z \sim \arcsin(2r/L - 1)$.

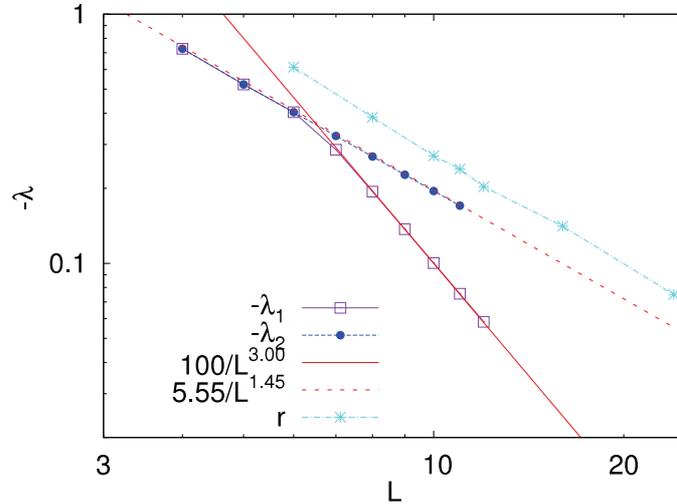


Figure 5. Gap of the superoperator \mathcal{L} , equation (2), for a one-spin bath (i.e., infinite temperature) from an exact calculation (squares and circles) and the relaxation rate of tDMRG simulations r (crosses). The largest nontrivial eigenvalue of \mathcal{L} scales as $\lambda_1 \sim -100/L^{3.0}$ (full line), while the second one goes as $\lambda_2 \sim -5.55/L^{1.45}$ (dashed line). The scaling of the tDMRG convergence rate r with L is the same as that of λ_2 .

3.3. Spectral properties of \mathcal{L}

The spectral properties of a Liouvillian superoperator \mathcal{L} , i.e., the linear operator representing the right-hand side of the master equation (2), are important for several reasons. For instance, they determine the rate of relaxation to NESS as well as deviations from NESS expectation values at finite times. The superoperator \mathcal{L} is non-Hermitian and therefore has a spectrum of eigenvalues $\lambda_k, k = 0, \dots, 4^L - 1$, lying in a complex plane. We shall order eigenvalues λ_k in a descending order according to their real part, starting with the largest, $\lambda_0 = 0$. The right eigenvector $|x_0^R\rangle$, corresponding to λ_0 , is the sought-for NESS state ρ , symbolically $|x_0^R\rangle = |\rho\rangle$. As a consequence of the trace preservation of \mathcal{L} the left eigenvector $\langle x_0^L|$ corresponding to λ_0 is on the other hand proportional to the identity operator $\sim \mathbb{I}$, irrespective of the system, $\langle x_0^L| = \langle \mathbb{I}|$. In our system, λ_0 is always nondegenerate. For simplicity we shall in this subsection discuss properties of the isotropic Heisenberg model with a one-spin bath, that is at an infinite temperature. The driving potential is weak, $\mu = 0.02$; however, the values of the eigenvalues are in the linear response regime largely independent of μ .

Besides $|\rho\rangle$, eigenvalues and eigenvectors that are closest to λ_0 (in real part) are also of interest. Because the dynamics governed by the Lindblad equation is contractive, all real parts of eigenvalues are non-positive, $\text{Re}(\lambda_k) \leq 0$. The linear operator \mathcal{L} is in general non-diagonalizable with the Jordan canonical form; see for instance [25] for a discussion of spectral decomposition in such case for quadratic fermionic systems. For the isotropic Heisenberg model (2) we have found by numerical computation that for a few eigenvalues with the largest real parts the number of linearly independent eigenvectors is always the

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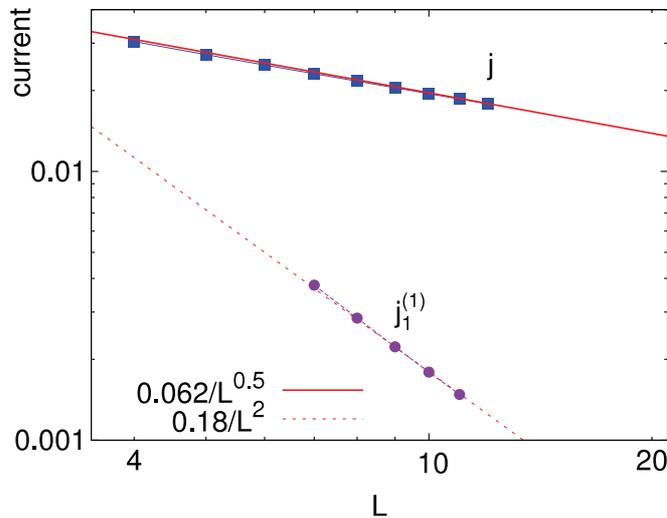


Figure 6. The current in the NESS state, j (squares; the same data as in figure 2), and the current for the second eigenvector, corresponding to λ_1 , at the first site (where it is the largest), $j_1^{(1)} = \langle j_1 | x_1^R \rangle$ —all obtained using exact diagonalization and a one-spin bath (i.e., $T = \infty$).

same as the multiplicity of the corresponding eigenvalue and therefore the Jordan form for these eigenvalues is trivial, of dimension 1. We can therefore write

$$\mathcal{L} = \lambda_0 |x_0^R\rangle\langle x_0^L| + \lambda_1 |x_1^R\rangle\langle x_1^L| + \lambda_2 |x_2^R\rangle\langle x_2^L| + \dots, \tag{10}$$

where left and right eigenvectors are mutually orthogonal, $\langle x_j^L | x_k^R \rangle = \delta_{jk}$, with the standard Hilbert–Schmidt inner product $\langle A | B \rangle = \text{tr}(A^\dagger B)$, and we normalize left eigenvectors $\langle x_j^L |$. Then the value of the real part of λ_1 , also called the gap of the Liouvillian, determines the rate of convergence with which the NESS is reached from $\rho(0)$, while the corresponding eigenvector gives the deviation of $\rho(t)$ from the NESS ρ . In addition, the scaling of the gap with the system size L can be used to locate nonequilibrium phase transitions. For studies of this phenomenon in quantum system see [26]; for classical systems see e.g. [27].

We have determined the lowest four eigenvalues $\lambda_{0,1,2,3}$ and their corresponding left and right eigenvectors using numerically exact diagonalization on small systems of size $L \leq 12$. In addition, we determined the relaxation rate r of our tDMRG solution $\rho(t)$ by fitting the convergence of the magnetization at the middle of chain to its asymptotic value, $\text{tr}(\rho(t)\sigma_{L/2}^z) - \langle \sigma_{L/2}^z \rangle \sim \exp(-rt)$ (the same r is also obtained by looking at the convergence of the magnetization current). In our isotropic Heisenberg chain with a one-spin bath, λ_1 and λ_2 always have zero imaginary part, $\text{Im}(\lambda_{1,2}) = 0$. The data in figure 5 show that the gap of \mathcal{L} decreases as $\lambda_1 \sim 1/L^3$. The same scaling with L is obtained also for the XX model [28]. The eigenvalue λ_1 is $2\times$ degenerate for $L \leq 6$, while it is nondegenerate for $L > 6$. On the other hand, λ_2 becomes $2\times$ degenerate for $L > 6$. The scaling of λ_2 for large L is $\lambda_2 \sim 1/L^{1.45}$ and therefore decays with L in a much slower way than λ_1 . What is interesting is that the convergence rate of the tDMRG simulation is not

given by λ_1 , but rather follows the scaling of λ_2 . This is so because the expectation values of the current and magnetization, which are relevant observables for our discussion of transport, are very small for the eigenvector corresponding to λ_1 . In fact, for $L \leq 6$ they are identically zero, while for larger L their values are shown in figure 6. One can see that the contribution from $|x_1^R\rangle$ to the magnetization current (and the magnetization as well) scales as $\sim 1/L^2$ and is indeed negligible in the thermodynamic limit². For instance, the relative contribution at $L = 128$ would be $(0.18/L^2)/(0.062/L^{0.5}) \approx 0.002$, which is below the precision of our tDMRG simulations. Namely, we estimate that the truncation errors in tDMRG simulations result in a relative error in the current below 1% at $T_{\text{meas}} = \infty$ and below 5% at $T_{\text{meas}} = 4.8$, both for the largest sizes shown.

4. Conclusion

Using extensive numerical calculations of nonequilibrium steady states close to equilibrium in chains of up to 256 spins we have shown that the diffusion constant of the magnetization in the isotropic Heisenberg model scales with the system length as $D \sim L^{0.5}$ for temperatures larger than the value of the exchange interaction. In this temperature regime the anomalous diffusion exponent of 0.5 seems largely independent of the temperature. The spectral properties of the Liouville superoperator have also been explored, showing that the gap scales as $\sim 1/L^3$ with the system length L , while the tDMRG expectation values of the magnetization and current converge in a shorter time, increasing as $\sim L^{1.45}$.

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² On the other hand, at a maximal one-spin driving, $\mu = 1$, the current in the NESS scales as $j \sim 1/L^2$; see [22].

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