

# Decoherence in regular systems

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## Abstract

We consider unitary evolution of finite bipartite quantum systems and study the time dependence of purity for initial cat states—coherent superpositions of Gaussian wavepackets. We derive an explicit formula for purity in systems with nonzero time averaged coupling, a typical situation for systems where an uncoupled part of the Hamiltonian is Liouville integrable. Previous analytical studies have been limited to harmonic oscillator systems, but with our theory we are able to derive analytical results for general integrable systems. Two regimes are clearly identified: at short times purity decays due to decoherence, whereas at longer times it decays because of relaxation.

**Keywords:** decoherence, purity, integrable systems, relaxation

## 1. Introduction

Quantum mechanics is a linear theory and so a superposition of solutions is also an admissible solution. The appearance of such superpositions at the quantum level and at the same time their absence from the macroscopic classical world has troubled scientists from the very beginning; an example is the famous ‘Schrödinger cat’ [1]. Their absence is usually explained in terms of *decoherence* due to the coupling with external degrees of freedom; see [2] for a review. In recent years the process of decoherence has actually been experimentally measured [3, 4]. In the present paper we will derive the decay of purity ( $1 -$  linear entropy) for initial cat states in finite Hamiltonian systems where an *uncoupled* part of the Hamiltonian generates regular (integrable) dynamics. Such systems are common among theoretical models (e.g. Jaynes–Cummings, ion trap quantum computer) and can within a certain approximation even be realized in experiments. Our results are relevant also for the decoherence of macroscopic superpositions in general systems if the decoherence is faster than any dynamical timescale involved. Note however that a strict decoherence in the mathematical sense, i.e. irreversible loss of coherence, is possible only in the thermodynamic limit. Still, ‘for all practical purposes’ one can have decoherence also in a sufficiently large finite system where the actual act of reversal is close to impossible due to the high sensitivity to perturbations. Almost all theoretical studies of decoherence start from a master equation for the reduced density matrix; see e.g. [5] for the case of a harmonic oscillator. Derivation of such a master equation is possible only for very simple

systems, for instance for a harmonic oscillator coupled to an infinite heat bath consisting of harmonic oscillators [6]. Our approach here is different. We do not use a master equation but rather start from first principles, i.e. from a Hamiltonian describing a system coupled to a finite ‘bath’. In the case of regular uncoupled dynamics we are able to derive the decay of purity for initial cat states. We stress that the result applies to any integrable dynamics, not just e.g. harmonic oscillators, and that the coupling can be quite arbitrary. Thus we will describe phenomena that go beyond simple ‘reversible decoherence’ discussed for instance for two coupled cavities [7]. Furthermore, our results might shed new light on the occurrence of decoherence in infinite-dimensional systems, e.g. a harmonic oscillator bath [6], by taking the appropriate limits.

## 2. Process of decoherence

Let us write the Hamiltonian of the entire system as

$$H = H_0 + \delta \cdot V, \quad H_0 = H_c \otimes \mathbb{1}_e + \mathbb{1}_c \otimes H_e, \quad (1)$$

where  $H_0$  is an uncoupled part of the Hamiltonian and  $V$  is the coupling, with  $\delta$  being its dimensionless strength. We will use subscripts ‘c’ and ‘e’ to denote the ‘central’ subsystem and ‘environment’ (‘environment’ is used just as a label for the part of the system we will trace over, without any connotations for its properties). In the present paper we study purity decay for the initial product state of the form

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|\psi_{c1}(0)\rangle + |\psi_{c2}(0)\rangle) \otimes |\psi_e(0)\rangle, \quad (2)$$

with all three states  $|\psi_{c1,2}(0)\rangle$  and  $|\psi_e(0)\rangle$  being localized Gaussian wavepackets. We assume that the initial state of the central system is composed of widely separated packets (i.e. is a cat state), so two constituent states are nearly orthogonal,  $\langle\psi_{c1}(0)|\psi_{c2}(0)\rangle \approx 0$ . However, manipulations with macroscopic superpositions are quite nontrivial. In experimental situations the coherence of widely separated packets (e.g.  $|\alpha\rangle + |\beta\rangle$  for large boson parameter  $\alpha$ ) might be washed out due to the presence of losses [8]. Purity is defined as

$$I(t) = \text{tr}_c \rho_c^2(t), \quad \rho_c(t) = \text{tr}_e |\psi(t)\rangle\langle\psi(t)|, \quad (3)$$

where  $|\psi(t)\rangle = \exp(-iHt/\hbar)|\psi(0)\rangle$ . For our cat state (2) the initial reduced density matrix reads

$$\rho_c(0) = \frac{1}{2}(|\psi_{c1}(0)\rangle\langle\psi_{c1}(0)| + |\psi_{c2}(0)\rangle\langle\psi_{c2}(0)| + |\psi_{c1}(0)\rangle\langle\psi_{c2}(0)| + |\psi_{c2}(0)\rangle\langle\psi_{c1}(0)|). \quad (4)$$

It has two diagonal terms and two off-diagonal terms, also called coherences. These off-diagonal matrix elements are characteristic for coherent quantum superpositions. The process of decoherence then causes the *decay* of off-diagonal matrix elements of  $\rho_c(t)$ , so after some characteristic decoherence time  $\tau_{\text{dec}}$  we end up with the reduced density matrix  $\rho_c^{\text{mix}}(t)$  which is a statistical *mixture* of two diagonal terms only:

$$\rho_c^{\text{mix}}(\tau_{\text{dec}}) = \frac{1}{2}(|\psi_{c1}(\tau_{\text{dec}})\rangle\langle\psi_{c1}(\tau_{\text{dec}})| + |\psi_{c2}(\tau_{\text{dec}})\rangle\langle\psi_{c2}(\tau_{\text{dec}})|). \quad (5)$$

The purity of this reduced density matrix is  $I(\tau_{\text{dec}}) = 1/2$ . Whereas the initial density matrix  $\rho_c(0)$  ( $I(0) = 1$ ) has no classical interpretation, the decohered density matrix  $\rho_c^{\text{mix}}(\tau_{\text{dec}})$  does have one. The aim of this paper is to explicitly show decoherence, i.e. to derive the transition

$$\rho_c(0) \xrightarrow{\tau_{\text{dec}}} \rho_c^{\text{mix}}(\tau_{\text{dec}}). \quad (6)$$

We are going to do this for systems with a regular uncoupled part  $H_0$  of the Hamiltonian. The progress of decoherence will be ‘monitored’ by calculating the purity (3) which will decay from the initial value 1 to 1/2 after  $\tau_{\text{dec}}$ . When speaking about the decay of off-diagonal matrix elements of  $\rho_c(t)$  we should be a little careful though as the notion of ‘off-diagonal’ depends on the basis. It has recently been shown [9, 10] that the purity in regular systems and for initial localized wavepackets decays on a very long timescale  $\tau_p \sim 1/\delta$ , which is independent of  $\hbar$ . Localized wavepackets are therefore very long-lived states—the pointer states (they get entangled very slowly), and this is a preferred basis which we have in mind when speaking about the off-diagonal matrix elements. We will explain the meaning of pointer states in more detail later, after we derive an analytic expression for the purity decay. Provided  $\tau_p$  is larger than  $\tau_{\text{dec}}$  we can assume that during the process of decoherence, propagation of initial constituent states still results in approximately product states, i.e. denoting  $|\psi_{1,2}(t)\rangle = \exp(-iHt/\hbar)|\psi_{c1,2}(0)\rangle \otimes |\psi_e(0)\rangle$ , we have for  $t < \tau_p$ ,  $|\psi_{1,2}(t)\rangle \approx |\psi_{c1,2}(t)\rangle \otimes |\psi_{e1,2}(t)\rangle$ . Note that we do not assume that  $|\psi_{c,e}(t)\rangle$  can be obtained by propagation with  $H_{c,e}$  alone; in fact they cannot be, because, for instance, the state of the environment  $|\psi_{e1,2}(t)\rangle$  depends on the state of the central system. The above product form of individual states

also justifies the form of the decohered density matrix  $\rho_c^{\text{mix}}$  (5) and is an important ingredient for a self-consistent explanation of the process of decoherence (6). The condition  $\tau_p > \tau_{\text{dec}}$  means that the decoherence timescale  $\tau_{\text{dec}}$  we are interested in is shorter than the relaxation timescale  $\tau_p$  on which individual pointer states relax to their equilibrium. From the result of this paper (15) we will see that this condition is satisfied provided the separation between two packets is large enough and/or  $\hbar$  is sufficiently small.

### 3. Purity decay

Let us now calculate off-diagonal matrix elements of  $\rho_c(t)$ . They are of the form

$$\text{tr}_e |\psi_1(t)\rangle\langle\psi_2(t)| \approx |\psi_{c1}(t)\rangle\langle\psi_{c2}(t)|\langle\psi_{e2}(t)|\psi_{e1}(t)\rangle. \quad (7)$$

Assuming the states of the central system are still approximately orthogonal,  $\langle\psi_{c1}(t)|\psi_{c2}(t)\rangle \approx 0$ , the purity is simply  $I(t) = \frac{1}{2}(1 + F_e(t))$ , with

$$F_e(t) = |\langle\psi_{e2}(t)|\psi_{e1}(t)\rangle|^2 = \text{tr}_e \rho_{e1}(t)\rho_{e2}(t). \quad (8)$$

Two reduced density matrices of the environment are  $\rho_{e1,2}(t) = \text{tr}_c |\psi_{1,2}(t)\rangle\langle\psi_{1,2}(t)|$ . The quantity  $F_e(t)$  gives the size of the off-diagonal matrix elements and its decay is an indicator of decoherence. It is an overlap on the environment subspace of two states at time  $t$  obtained by the same evolution from two different initial product states. It is similar to the fidelity [11], which is the overlap of two states obtained from the same initial condition under two different evolutions. In fact  $F_e(t)$  can be connected with a quantity called the reduced fidelity [12], i.e. the fidelity on the subspace; see also [13] for a connection between decoherence and fidelity. Here we will not use this analogy as we will calculate  $F_e(t)$  directly. Calculating  $F_e(t)$  is easier in the interaction picture, given by  $|\psi_{1,2}^M(t)\rangle = M(t)|\psi_{1,2}(0)\rangle$ , with  $M(t) = \exp(iH_0t/\hbar)\exp(-iHt/\hbar)$  being the so-called echo operator used extensively in the theory of fidelity decay. Since  $\exp(-iH_0t/\hbar)$  factorizes, using  $\rho_{e1,2}^M(t)$  instead of  $\rho_{e1,2}(t)$  will not change  $F_e(t)$ . For systems where there exists an averaging time  $t_{\text{ave}}$  after which a time average of the coupling in the interaction picture,  $V(t) = \exp(iH_0t/\hbar)V\exp(-iH_0t/\hbar)$ , converges,

$$\bar{V} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt V(t), \quad (9)$$

we can further simplify the echo operator. Note that  $t_{\text{ave}}$  is a classical time if the classical limit exists and the effective  $\hbar$  is sufficiently small, i.e.  $t_{\text{ave}}$  does not depend either on  $\delta$  or on  $\hbar$ . It is given by the classical correlation time of the coupling  $V(t)$ . Nontrivial  $\bar{V}$  will typically occur in regular (integrable) systems. But note that only  $H_0$  needs to be regular, whereas the coupling  $V$  (and therefore also  $H$ ) can be arbitrary. For  $\delta \ll 1$  one can show that the leading order expression (in  $\delta$ ) for the echo operator is

$$M(t) = e^{-i\delta t \bar{V}/\hbar}. \quad (10)$$

For details see e.g. [10]. We proceed with a semiclassical evaluation of  $F_e(t)$ . The average  $\bar{V}$  is by construction

a function of action variables only and therefore the semiclassical evaluation of  $F_c(t)$  is simplified. The calculation goes along the same lines as for the evaluation of fidelity [11] and of the purity for individual coherent initial states [10]. A sum over quantum numbers is replaced with an integral over the action space and quantum operator  $\bar{V}$  is replaced with its classical limit  $\bar{v}(\mathbf{j})$ , where  $\mathbf{j} = (\mathbf{j}_c, \mathbf{j}_e)$  is a vector of actions having  $d = d_c + d_e$  components if the central system and environment have  $d_c$  and  $d_e$  degrees of freedom, respectively. Let us consider Gaussian packets  $\langle \mathbf{j}_a | \psi_a \rangle \propto \exp\{-i(\mathbf{j}_a - \mathbf{j}_a^*) \cdot \Lambda_a(\mathbf{j}_a - \mathbf{j}_a^*)/2\hbar + i\theta_a^* \cdot \mathbf{j}_a/\hbar\}$ , centred at  $(\mathbf{j}_a^*, \theta_a^*)$  where  $\Lambda_a$  is a positive squeezing matrix determining its shape. The semiclassical expression of the initial density  $p_a(\mathbf{j}_a) = \langle \mathbf{j}_a | \rho_a(0) | \mathbf{j}_a \rangle$  reads

$$p_a(\mathbf{j}_a) = \left(\frac{\hbar}{\pi}\right)^{d_a/2} \sqrt{\det \Lambda_a} \exp\{-i(\mathbf{j}_a - \mathbf{j}_a^*) \cdot \Lambda_a(\mathbf{j}_a - \mathbf{j}_a^*)/\hbar\}. \quad (11)$$

Subscript ‘a’ takes the values ‘c1’, ‘c2’ and ‘e’, for the three initial packets constituting the initial state (2). Writing for short  $p_{1,2}(\mathbf{j}) = p_{c1,2}(\mathbf{j}_c)p_e(\mathbf{j}_e)$ , we have the following expression for  $F_c(t)$ :

$$F_c(t) = \hbar^{-2d} \int d\mathbf{j} d\tilde{\mathbf{j}} \exp\left(-i\frac{\delta t}{\hbar}\Phi\right) p_1(\mathbf{j}) p_2(\tilde{\mathbf{j}}), \quad (12)$$

$$\Phi = \bar{v}(\mathbf{j}_c, \mathbf{j}_e) - \bar{v}(\tilde{\mathbf{j}}_c, \mathbf{j}_e) + \bar{v}(\tilde{\mathbf{j}}_c, \tilde{\mathbf{j}}_e) - \bar{v}(\mathbf{j}_c, \tilde{\mathbf{j}}_e).$$

Note that the above integral is essentially a classical average (see [14] for some results) over two densities corresponding to two states  $|\psi_{1,2}(0)\rangle$  and is therefore a sort of cross-correlation function. We expand the phase around the position  $\mathbf{j}_e^*$  of the environmental state,  $\Phi \approx [\bar{v}'_e(\mathbf{j}_c) - \bar{v}'_e(\tilde{\mathbf{j}}_c)](\mathbf{j}_e - \mathbf{j}_e^*) - [\bar{v}'_e(\mathbf{j}_c) - \bar{v}'_e(\tilde{\mathbf{j}}_c)](\tilde{\mathbf{j}}_e - \mathbf{j}_e^*)$ , where  $\bar{v}'_e(\mathbf{j}_c) = \partial \bar{v}(\mathbf{j}_c, \mathbf{j}_e^*)/\partial \mathbf{j}_c$  is a vector of partial derivatives with respect to the environment, evaluated at the position of the environmental packet  $\mathbf{j}_e^*$ . Integration over the action variables now gives

$$F_c(t) = \exp(-t^2/\tau_{\text{dec}}^2), \quad \tau_{\text{dec}} = \sqrt{\hbar/C_e}/\delta \quad (13)$$

$$C_e = \frac{1}{2}(\bar{v}'_e(\mathbf{j}_{c1}^*) - \bar{v}'_e(\mathbf{j}_{c2}^*))\Lambda_e^{-1}(\bar{v}'_e(\mathbf{j}_{c1}^*) - \bar{v}'_e(\mathbf{j}_{c2}^*)).$$

We can see that the decay time  $\tau_{\text{dec}}$  for the cat state is indeed smaller than the decay time for individual states  $\tau_p \sim 1/\delta$ , provided  $\hbar \ll C_e$ . In the linear response regime  $C_e$  is just the semiclassical expression for

$$C_e = \frac{1}{\hbar}(\langle [\bar{V} - \langle \bar{V} \rangle_e]^2 \rangle_1 + \langle [\bar{V} - \langle \bar{V} \rangle_e]^2 \rangle_2 + 2\langle \bar{V} \rangle_1 \langle \bar{V} \rangle_2 - \langle \langle \bar{V} \rangle_{c1} \langle \bar{V} \rangle_{c2} \rangle_e - \langle \langle \bar{V} \rangle_{c2} \langle \bar{V} \rangle_{c1} \rangle_e), \quad (14)$$

where the subscripts denote which initial state the average is performed with respect to. Plugging  $F_c(t)$  (13) into the expression for purity  $I(t)$  we get the purity decay for times  $t < \tau_p$ . We can actually calculate the purity for longer times as well. That is, after  $\tau_{\text{dec}}$  the reduced density matrix is a statistical mixture of states  $|\psi_{c1}(t)\rangle$  and  $|\psi_{c2}(t)\rangle$ . The purity can then be written as the sum of purities for individual states as there are no quantum coherences present any longer. Therefore, a complete formula for purity valid also for longer times is

$$I(t) = \frac{1}{4}(I_1(t) + I_2(t) + 2F_c(t)), \quad (15)$$

where  $I_{1,2}(t) = \text{tr}_c \rho_{c1,2}^2(t)$  are purities of individual states.  $I_{1,2}(t)$  decay on a long timescale  $\tau_p$  and have been calculated in [10]. Here we will just list the result:

$$I_{1,2}(t) = \frac{1}{\sqrt{\det(\mathbb{I} + (\delta t)^2 u_{1,2})}}, \quad u_{1,2} = \Lambda_{c1,2}^{-1} \bar{v}''_{ce} \Lambda_e^{-1} \bar{v}''_{ce}, \quad (16)$$

with a  $d_c \times d_e$  dimensional matrix  $\bar{v}''_{ce}$  of second derivatives evaluated at the position  $\mathbf{j}_{c1}^*$ , or  $\mathbf{j}_{c2}^*$ , of the initial state,  $(\bar{v}''_{ce})_{kl} = \partial^2 \bar{v}(\mathbf{j}_{1,2}^*)/\partial(\mathbf{j}_c)_k \partial(\mathbf{j}_e)_l$ .

The formula for  $I(t)$  (15) is our main result. Let us discuss it more in detail. First, there are two timescales: a short one on which  $F_c(t)$  decays and a long one on which  $I_{1,2}(t)$  decay. On a short timescale  $\tau_{\text{dec}}$  purity drops to  $I(\tau_{\text{dec}}) = 1/2$ , signalling decoherence. This decay of  $I(t)$  has a Gaussian form (13) and is generally the faster the further apart the centres of two initial states,  $|\psi_{c1}(0)\rangle$  and  $|\psi_{c2}(0)\rangle$ , are. Expanding  $\bar{v}'_e(\mathbf{j}_{c2}^*)$  around the position  $\mathbf{j}_{c1}^*$  we do indeed have  $C_e \approx \frac{1}{2}[\bar{v}''_{ce} \cdot (\mathbf{j}_{c2}^* - \mathbf{j}_{c1}^*)]\Lambda_e^{-1}[\bar{v}''_{ce} \cdot (\mathbf{j}_{c2}^* - \mathbf{j}_{c1}^*)]$ . If the coupling  $\bar{v}$  is between all pairs of degrees of freedom we have  $C_e \propto d_c d_e (\mathbf{j}_{c2}^* - \mathbf{j}_{c1}^*)^2$ . The decay time  $\tau_{\text{dec}}$  (13) is therefore inversely proportional to the distance between the packets  $|\mathbf{j}_{c2}^* - \mathbf{j}_{c1}^*|$  and to  $\sqrt{d_e}$ . Similar scaling of the decoherence time with the number of degrees of freedom has been recently experimentally measured in a NMR context [15]. Note that the decoherence time decreases, i.e. we have  $\hbar \ll C_e$ , also when we approach the semiclassical limit  $\hbar \rightarrow 0$ . As the effective  $\hbar$  determines the dimensionality of the relevant Hilbert space,  $\tau_{\text{dec}}$  will be the smaller the larger the Hilbert space we have. It also decreases with the number of degrees of freedom  $d_e$  of the environment, meaning that decoherence is faster for an environment with more degrees of freedom. Taking the thermodynamic limit is not straightforward though. The fact that the decoherence time  $\tau_{\text{dec}}$  goes to zero if the number of degrees of freedom  $d_e$  of the environment is increased at constant coupling  $\delta$  is not a surprise. Indeed, if we have a coupling of the same strength with infinitely many (and infinitely fast) degrees of freedom, decoherence will occur instantly. To have a physical thermodynamical limit one has to take  $d_e \rightarrow \infty$  and at the same time decrease the coupling strength to higher modes or make the so-called ‘ultraviolet cut-off’ as in e.g. the Caldeira–Leggett model [6]. In such a case one would get a finite  $\tau_{\text{dec}}$ . For a self-consistent description using  $I(t)$  (15) one also needs  $\tau_{\text{dec}} \gg t_{\text{ave}}$ . Note also that decoherence has a Gaussian shape, in contrast to the case for the master equation approach, where the decay is exponential. A Gaussian form of decoherence has been obtained for macroscopic superpositions in [16].

Of course, all this holds provided the average perturbation  $\bar{v}$  is nonzero. An interesting suppression of the decoherence might arise for  $\bar{v} = 0$ , for which extreme stability of the quantum fidelity has recently been found [17]. After the decoherence time  $\tau_{\text{dec}}$  we are left with a statistical mixture of states, and further decay of the purity  $I(t)$ , due to the decay of  $I_{1,2}(t)$ , is a consequence of *relaxation* to the final state. This relaxation happens on a long timescale  $\tau_p$  given by the slower decaying  $I_{1,2}(t)$ ; see [10] for details. In order to illustrate the decay of  $I(t)$  we performed numerical simulations.

#### 4. Numerical example

We consider two coupled anharmonic oscillators, i.e.  $d_c = 1$ , with the Hamiltonian

$$H = \gamma_c(\hbar a_c^+ a_c - \Delta)^2 + \gamma_e(\hbar a_e^+ a_e - \Delta)^2 + \delta \hbar^2 (a_c^+ + a_e)^2 (a_e^+ + a_c)^2 \quad (17)$$

where  $a_a^+$ ,  $a_a$  denote boson raising/lowering operators. All initial wavepackets are boson coherent states,  $|\psi_{c,e}(0)\rangle = e^{\alpha a^+ - \alpha^* a}|0\rangle$ , where  $|0\rangle$  is the ground state. The parameter  $\alpha$  is chosen as  $\alpha = \sqrt{j^*/\hbar}$  with the initial action  $j_e^* = 0.1$  for the environment and  $j_{c1}^* = 0.2$  and  $j_{c2}^* = 0.01$  for the two states of the central system. The ‘squeezing parameter’  $\Lambda$  (11) is for coherent states equal to  $\Lambda = 1/(2j^*)$ . Other parameters of the Hamiltonian are  $\gamma_c = 1$ ,  $\gamma_e = 0.6456$  and  $\Delta = 1.2$ . The coupling strength is set to  $\delta = 0.01$ , but note that our theory is often not limited to small  $\delta$  [10]. The time averaged coupling is easily calculated from the classical limit of the Hamiltonian and is  $\bar{v} = 4j_c j_e$ . Using this we easily evaluate  $C_e$  (13):

$$C_e = 16j_e^*(j_{c1}^* - j_{c2}^*)^2 \approx 0.058, \quad (18)$$

and the decay time  $\tau_{\text{dec}}$  of  $F_e(t)$  (13) is

$$\tau_{\text{dec}} = \sqrt{\hbar/j_e^*}/(4\delta|j_{c1}^* - j_{c2}^*|) \approx 416\sqrt{\hbar}. \quad (19)$$

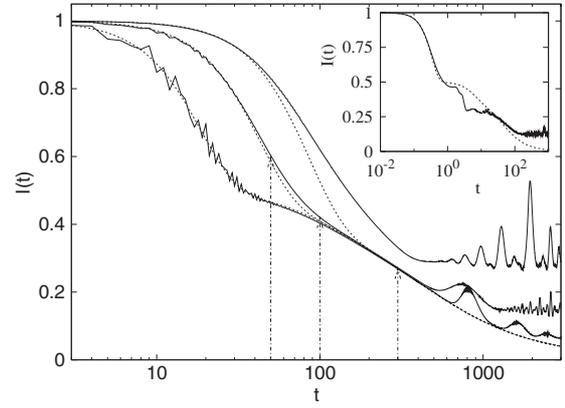
The matrix  $u$  needed in  $I_{1,2}(t)$  (16) is just a number equal to  $u_{1,2} = 64j_e^* j_{c1,2}^*$ . The theoretical decay of  $I_{1,2}(t)$  (16) is therefore

$$I_{1,2}(t) = \frac{1}{\sqrt{1 + (t/\tau_{p1,2})^2}}, \quad \tau_{p1,2} = \frac{1}{8\delta\sqrt{j_{c1,2}^* j_e^*}}. \quad (20)$$

In figure 1 we show the results of numerical simulation for three different values of  $\hbar$ . For the smaller values of  $\hbar$  we can clearly see the two regimes discussed. Initially  $I(t)$  decays due to decoherence as described by  $F_e(t)$ . After the decoherence time  $\tau_{\text{dec}}$ , i.e. after  $I(t)$  falls to the value  $1/2$ , relaxation begins. This regime is described by two decaying purities  $I_{1,2}(t)$  of individual states and does not depend on  $\hbar$ .

For the largest value of  $\hbar = 1/25$  finite-size effects can be observed. The origin of finite-size effects is twofold: first, the condition of a well separated decoherence timescale  $\tau_{\text{dec}}$  ( $\hbar$ -dependent) and relaxation timescale  $\tau_p$  ( $\hbar$ -independent) results in  $\hbar \ll C_e$ . Second, there is a finite saturation value of purity for nonvanishing  $\hbar$ ,  $I(t \rightarrow \infty) \sim \hbar^{d_c}$ . For the largest  $\hbar = 1/25$  shown in figure 1, the saturation value is  $I \sim 0.3$  which is almost as large as the purity at the end of decoherence,  $I(\tau_{\text{dec}}) = 1/2$ . Therefore this finite saturation value will start to influence the purity decay already at  $\tau_{\text{dec}} \approx 80$  and no relaxation can be observed.

In figure 2 we show a Wigner function of the reduced density matrix  $\rho_c^M(t) = \text{tr}_e M(t)\rho(0)M^\dagger(t)$  at times  $t = 0, 50, 100$  and  $300$ . The integral of the square of the Wigner function equals the purity. At  $t = 0$  we see strong oscillations in the Wigner function at the mid-point between two packets. This is a characteristic feature of coherent superpositions. At  $t = 50$  we are in the middle of decoherence where the oscillations and negative values of the Wigner function have already been reduced. At  $t = 100$  the process of decoherence has ended, visible as the absence of negative regions in the Wigner function. At still longer time  $t = 300$  the relaxation

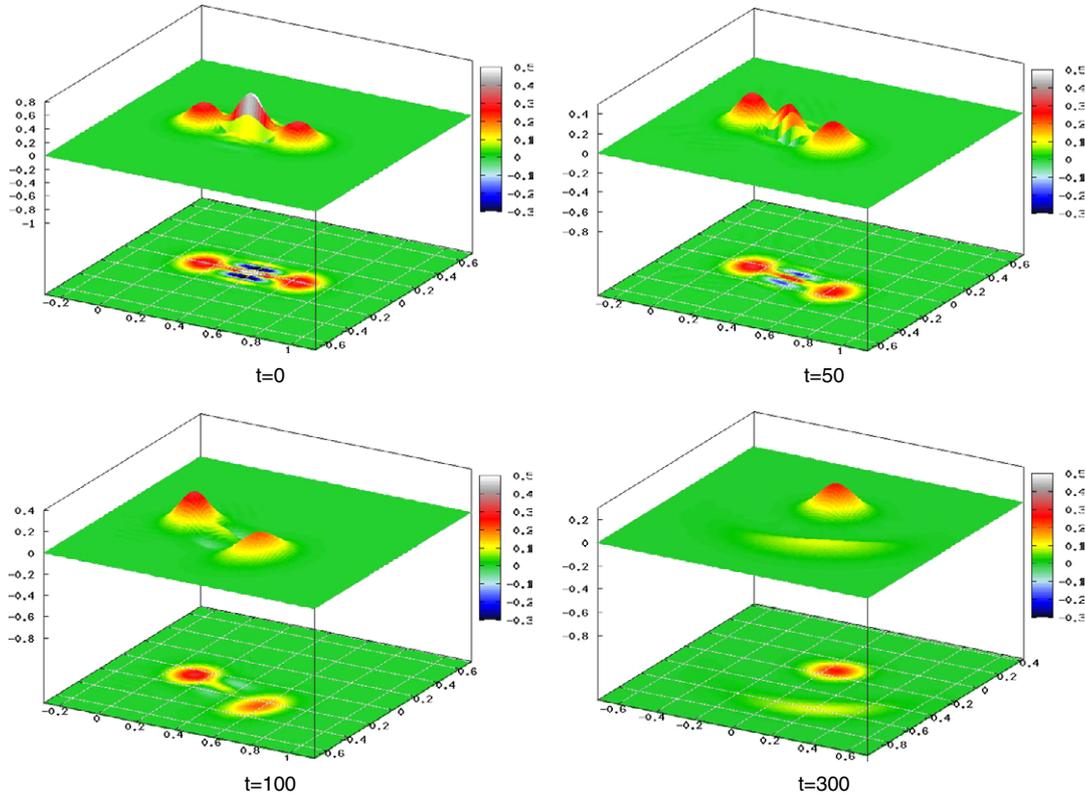


**Figure 1.** Purity decay for system (17) for an initial cat state and different  $1/\hbar = 25, 100$  and  $500$ : full curves from right to left. The dotted curves converging to a single curve for  $t > 100$  are the theoretical prediction for  $I(t)$  (15) using decay times (19), (20). The vertical arrows indicate times at which we show Wigner functions in figure 2. In the inset we show the decay for macrosuperposition; see the text for details.

is in full swing as the second packet has almost entirely disappeared. In figure 2 one can observe that the packets rotate around the origin and that the relaxation rates of the two packets are not equal. This can be explained by the form of the time averaged Hamiltonian in the interaction (i.e. echo) picture,  $\bar{v} = 4j_c j_e$ . This Hamiltonian causes rotation of the angle of the central system,  $\dot{\theta}_c = 4j_e$ . Using the value of  $j_e^* = 0.1$  we get that the packets should rotate for  $90^\circ$  around the origin in time  $t \approx 400$ , which agrees with the data shown. Unequal relaxation can be simply understood from the expression for  $\tau_{p1,2}$  (20). The packet with smaller  $j_c^*$  will relax on a longer timescale. This slower decay is a second-order effect due to smaller size of the packet in the action direction (i.e. larger  $\Lambda$ ) which causes a slower dephasing of angles  $\theta_e$  and in turn slower decay of the purity  $I(t)$ .

We should mention that at very large times (i.e. much later after the relaxation) we get revivals of purity. These are a simple consequence of having a finite number of eigenmodes of the uncoupled system for not too large  $\hbar$ , which causes a complex beating-like phenomenon at large times. Note that these revivals are not simple Rabi oscillations between the two subsystems, as discussed for instance for cavities in [7]. For our data (not shown in the figure) and times  $t < 10000$  we have a revival  $I \approx 0.95$  at  $t \approx 4000$  for  $\hbar = 1/25$ , a revival of  $I \approx 0.5$  at  $t \approx 8000$  for  $\hbar = 1/100$  and there is no revival for  $\hbar = 1/500$  (and  $t < 5000$ ). The revivals are therefore less prominent and happen at larger times for small  $\hbar$ . The same would happen if the central system had more than one degree of freedom. For systems with many degrees of freedom, high sensitivity to perturbations will also effectively prevent the possibility of restoring coherence upon time reversal.

Finally let us comment on the relevance of our results for the appearance of the macroworld. It has been pointed out [16] that the decoherence time for sufficiently macroscopic superpositions gets very short. Therefore, the system’s dynamics can be considered to be regular on this very short timescale, but on the other hand we need  $\tau_{\text{dec}} \gg t_{\text{ave}}$  for our theory to apply ( $t_{\text{ave}}$  does not depend on  $\delta$  or  $\hbar$ ). Resolution



**Figure 2.** Wigner functions for the reduced density matrix  $\rho_c^M(t)$  and initial cat state (2). The Planck constant is  $\hbar = 1/100$  and all other data are the same as for figure 1. On the  $x$  and  $y$  axes we plot  $\sqrt{2\hbar} \operatorname{Re} \alpha$  and  $\sqrt{2\hbar} \operatorname{Im} \alpha$ , respectively. Disappearance of the oscillations is an indicator of decoherence, whereas the fading of packets is a signature of relaxation. See the text for details.

of this problem is immediate if one looks at the quantum expression for  $C_c$  (14). For short times one has to replace  $\bar{V}$  with an ‘instantaneous’ operator  $V(0)$  and evaluate quantum expectation values using classical averaging. Doing this for our model (17) for instance, we get  $\tau_{\text{dec}}(t \ll t_{\text{ave}}) = \tau_{\text{dec}}/4$ , with  $\tau_{\text{dec}}$  given in (19). This theoretical prediction agrees with the numerical curve for  $\delta = 0.06$ ,  $\hbar = 1/100$  and  $j_{c2}^* = 1$ , shown in the inset of figure 1, where  $\tau_{\text{dec}}$  is much shorter than for the three curves in the main plot. Again, the theoretical curve (dotted) agrees with the numerics (full line). Our theory therefore covers the regime  $\tau_{\text{dec}} \gg t_{\text{ave}}$  as well as  $\tau_{\text{dec}} \ll t_{\text{ave}}$ .

## 5. Conclusions

We have derived an analytic semiclassical expression for the purity decay of initial cat states in finite composite systems which are integrable in the absence of coupling and have a nonzero time averaged coupling. Note that the coupling can in general break integrability. Such systems are found both among theoretical models and in experiments. Pointer states are identified as localized action packets in the action components for which the time averaged coupling is nontrivial. The purity for superpositions of such pointer states first decays on a short timescale as a Gaussian, indicating decoherence. On a longer timescale, after the decoherence, it decays in an algebraic way, signifying relaxation to equilibrium. The theoretical results are confirmed by the numerical simulation of two coupled nonlinear oscillators.

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