Eigenvalue Statistics as an Indicator of Integrability of Nonequilibrium Density Operators

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We propose to quantify the complexity of nonequilibrium steady state density operators, as well as of long-lived Liouvillian decay modes, in terms of the level spacing distribution of their spectra. Based on extensive numerical studies in a variety of models, some solvable and some unsolved, we conjecture that the integrability of density operators (e.g., the existence of an algebraic procedure for their construction in finitely many steps) is signaled by a Poissonian level statistics, whereas in the generic nonintegrable cases one finds level statistics of a Gaussian unitary ensemble of random matrices. Eigenvalue statistics can therefore be used as an efficient tool to identify integrable quantum nonequilibrium systems.

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Introduction.—Random matrix theory (RMT) [1] is one of the most abstract yet successful models of statistical physics that is capable of universally describing such diverse phenomena in nature and society as quantum chromodynamics [2] and stock exchange volatility [3]. In general terms, RMT characterizes universal features of a certain phenomenon based on statistical correlations between the eigenvalues of a Hermitian matrix that describes the problem, be it the system Hamiltonian in a typical state basis or the covariance matrix of stocks in a portfolio. RMT then explains these eigenvalue correlations in terms of those of a probabilistic ensemble of random Hermitian matrices.

The so-called quantum chaos conjecture (QCC) [4–6] provided a deep connection between the eigenvalue correlations of quantum Hamiltonians of nonlinear single- (or few-) particle problems and the algorithmic complexity of the underlying classical trajectories. Namely, it has been shown [7] that dynamics where all classical trajectories are chaotic, i.e., exponentially unstable, results in a RMT spectral fluctuation of the corresponding quantum Hamiltonian. For Liouville integrable systems on the other side, following the argument by Berry and Tabor [8], the existence of a complete set of integrals of motion resulting in a full set of quantum numbers prohibits any statistical correlations in the quantum spectra and renders the corresponding level statistics Poissonian. Similarly, based on observations [9] it has been suggested that simple many-body quantum Hamiltonians that do not have classical limits possess Poissonian or RMT level statistics whenever they are integrable or strongly nonintegrable, respectively. As there is no systematic algorithmic method by which one can establish whether a certain system is integrable, i.e., exactly solvable or not, the level statistics has become a standard empirical indicator of integrability. It has been corroborated by a vast amount of numerical and experimental data [10].

The QCC describes the situation of closed quantum systems. In equilibrium, the density operator is given by the Gibbsian $\rho_{\rm eq} = Z^{-1}\exp(-\beta H)$, i.e., a mixture of eigenstates of the Hamiltonian H, $H|E_n\rangle = E_n|E_n\rangle$, with probabilities $p_n = Z^{-1}e^{-\beta E_n}$. Since a smooth, monotonic transformation $\rho \to \log\rho$ does not change the local level correlations, one can rephrase the old problem of level statistics for closed system Hamiltonians in terms of level statistics of the corresponding equilibrium density operator and formulate the QCC for $\rho_{\rm eq}$.

In open quantum systems, however, the evolution of the density operator is given in terms of a master equation with the Liouvillian generator that contains both the Hamiltonian and the dissipative terms, the latter coming from the interaction between the system and the environment. Within the Markovian approximation such evolution is given in terms of the Lindblad equation [11]

$$\frac{d}{dt}\rho = \hat{\mathcal{L}}\rho := -i[H,\rho] + \hat{\mathcal{D}}\rho, \tag{1}$$

with $\hat{\mathcal{D}}\rho := \sum_\mu 2L_\mu \rho L_\mu^\dagger - \{L_\mu^\dagger L_\mu, \rho\}$ being the quantum dissipation fully specified by a set of quantum-jump (Lindblad) operators L_μ. The positive semidefinite Hermitian operator $\rho(t)$ describes the quantum relaxation process from some initial state $\rho(0)$ to the steady state $\rho_0 = \rho(t\to\infty)$, satisfying $\hat{\mathcal{L}}\rho_0 = 0$.

In this Letter we formulate the QCC for nonequilibrium density operators, say for the nonequilibrium steady state (NESS) ρ_0 or even Hermitian decay modes (HDMs), i.e., right eigenoperators of $\hat{\mathcal{L}}$ with real [12] eigenvalues Λ_m, $\hat{\mathcal{L}}\rho_m = \Lambda_m\rho_m$, where $\Lambda_0 = 0$. We consider level statistics of the NESS and HDMs for several models of open quantum spin chains with boundary Lindblad driving and find, quite remarkably, that the former is Poissonian for all interacting and noninteracting cases that are exactly solvable, i.e., for which we can write ρ_0 explicitly in terms of a matrix product ansatz.

For models for which already the bulk Hamiltonian H is nonintegrable we find, consistently, that level statistics of the NESS and HDMs is described by a Gaussian unitary ensemble (GUE) of complex Hermitian random matrices (due to lack of time-reversal symmetry in generic nonequilibrium situations). We find GUE level statistics also for several models with integrable H but for which dissipative boundary conditions break integrability [13]. This leads us to a generalization of the QCC to nonequilibrium density operators.

In the last 30 years many solvable master equations describing classical nonequilibrium models have been discovered, in particular among lattice gas models [15]. Exactly solvable quantum many-body master equations though are only beginning to emerge, with so far only a handful of examples, namely, quasifree (quadratic) fermionic [16–18], or bosonic [19], systems with linear, or Hermitian quadratic [20–23], noise (Lindblad) operators, and maximally boundary driven XXZ chains [24]. One of the main difficulties is in the first place identifying promising candidates of solvable quantum master equations. The criterion suggested in this Letter, namely the generalized QCC, could be found very useful in this respect. For instance, in our study we find Possonian level statistics for the XXZ spin $1/2$ chain at large anisotropy Δ, indicating a possibility of a yet unknown exact solution for the NESS in the asymptotic regime $|\Delta| \gg 1$. This could be particularly interesting as the model exhibits diffusive spin transport in this regime [25–27].

The models and the method.—We shall demonstrate our conjecture on a number of one-dimensional spin $1/2$ systems that are driven at chain boundaries and optionally exhibit a bulk dephasing. All can be described by the XXZ type of Hamiltonian $H = \sum_{j=1}^{n-1}(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z) + \sum_{j=1}^{n} b_j\sigma_j^z$ for a chain of n sites. The dissipator $\hat{\mathcal{D}} = \hat{\mathcal{D}}^{\mathrm{driv}} + \gamma\hat{\mathcal{D}}^{\mathrm{deph}}$ is composed of a driving part $\hat{\mathcal{D}}^{\mathrm{driv}}$ that acts on the first and the last spin and is described by four local [28] Lindblad operators $L_1 = \sqrt{\Gamma(1-\mu+\bar{\mu})}\sigma_1^+$, $L_2 = \sqrt{\Gamma(1+\mu-\bar{\mu})}\sigma_1^-$ at the left end and $L_3 = \sqrt{\Gamma(1+\mu+\bar{\mu})}\sigma_n^+$, $L_4 = \sqrt{\Gamma(1-\mu-\bar{\mu})}\sigma_n^-$ at the right end, and of a dephasing $\gamma\hat{\mathcal{D}}^{\mathrm{deph}}$ described by one Lindblad operator at each site $L_j^{\mathrm{deph}} = (1/\sqrt{2})\sigma_j^z$, $j = 1, \ldots, n$. Relevant XXZ chain parameters are the anisotropy Δ and the external magnetic field b_j. For a homogeneous field the system is integrable [30], with the anisotropy changing the magnetization transport properties from ballistic for $|\Delta| < 1$ to diffusive for $\Delta > 1$ (in the absence of the field $b_j \equiv 0$). A staggered magnetic field renders the system quantum chaotic [31]. Dissipative parameters, describing the influence of environmental degrees of freedom, are the dephasing strength γ (a nonzero value causes the system to become diffusive), the coupling strength Γ (its precise value is inessential), and two driving parameters: the driving strength μ that determines how far from equilibrium we are ($\mu = 0$ causes an infinite-temperature equilibrium state) and $\bar{\mu}$ that determines the average magnetization.

The above class of systems includes solvable as well as nonsolvable out-of-equilibrium models. We shall first study spectral statistics of NESSs and at the end consider also HDMs. In each case we calculate the NESS ρ_0 (or a HDM) numerically exactly by using either an explicit solution, if it is known, or, by numerically finding the eigenvector of the Liouvillian $\hat{\mathcal{L}}$ using the Arnoldi method. For verifying the generalized QCC we have to assess as large a system as possible. The exponentially growing dimension of ρ_0 limits us to about $n = 20$ sites for solvable models; for nonsolvable systems the limiting factor is actually not the diagonalization of ρ_0 (being of dimension 2^n) but rather solving for the NESS, $\hat{\mathcal{L}}\rho_0 = 0$ (a set of 4^n linear equations). In all systems studied the total magnetization $Z = \sum_{j=1}^{n}\sigma_j^z$ is a constant of motion, i.e., $U_Z\hat{\mathcal{L}}(\rho)U_Z^\dagger = \hat{\mathcal{L}}(U_Z\rho U_Z^\dagger)$ for $U_Z = e^{-i\alpha Z}$. For spectral analysis we consider a block of dimension $\binom{n}{Z}$ of ρ_m with a fixed Z ($m = 0$ for the NESS and $m \geq 1$ for the HDM) and compute its unfolded [32] spectrum $\{\lambda_j\}$. Spectral statistics is then characterized by the level spacing distribution (LSD)—a histogram $p(s)$ of level spacings $s = \lambda_{j+1} - \lambda_j$, and compared to a Poissonian model of uncorrelated levels $p_{\mathrm{poisson}}(s) = \exp(-s)$ or the Wigner surmise of the GUE $p_{\mathrm{GUE}}(s) = (32/\pi^2)s^2\exp(-(4/\pi)s^2)$ [1].

Solvable open spin chains.—We study three instances of qualitatively different nonequilibrium solvable systems, a quadratic noninteracting one, a nonquadratic noninteracting one, and an interacting system, thus enabling us to explore a full range of complexity of solvable nonequilibrium systems.

Perhaps the simplest solvable nonequilibrium model is a boundary driven XX spin chain without dephasing ($\Delta = 0$, $b_j = 0$, $\gamma = 0$). Using a Jordan-Wigner transformation the Liouvillian becomes quadratic in fermionic operators and can be readily diagonalized [16]. The system is ballistic due to a noninteraction of fermionic normal modes. We calculate the NESS using a compact matrix product operator form of ρ_0 with matrices of fixed dimension 4 [33]. For $\bar{\mu} = 0$ the nonequilibrium XX chain has a parity symmetry $P = XR$, where $X = \prod_{j=1}^{n}\sigma_j^x$ and R is a left-right reflection $R|s_1, s_2, \ldots, s_n\rangle := |s_n, \ldots, s_2, s_1\rangle$, $s_j \in \{\uparrow, \downarrow\}$, as well as an additional antiunitary symmetry $T = Z_2K$, where K is a complex conjugation and $Z_2 = \prod_{j=1}^{n/2}\sigma_{2j}^z$. To remove these two symmetries we use a nonzero $\bar{\mu} = 0.3$. In Fig. 1(a) the LSD is shown; small deviations from Poissonian statistics can be attributed to finite-size effects.

The next solvable model that we consider is the XX chain ($\Delta = 0$, $b_j = 0$) with nonzero dephasing for which the system becomes diffusive. The dephasing term $\hat{\mathcal{L}}^{\mathrm{deph}}$ is quartic in fermionic operators; nevertheless, the NESS can be explicitly written [22] in powers of the driving μ due to a closing hierarchy of correlation functions [23]. Nonzero dephasing removes the antiunitary symmetry T while nonzero $\bar{\mu} = 0.3$ breaks the parity P. We can see in Fig. 1(b)

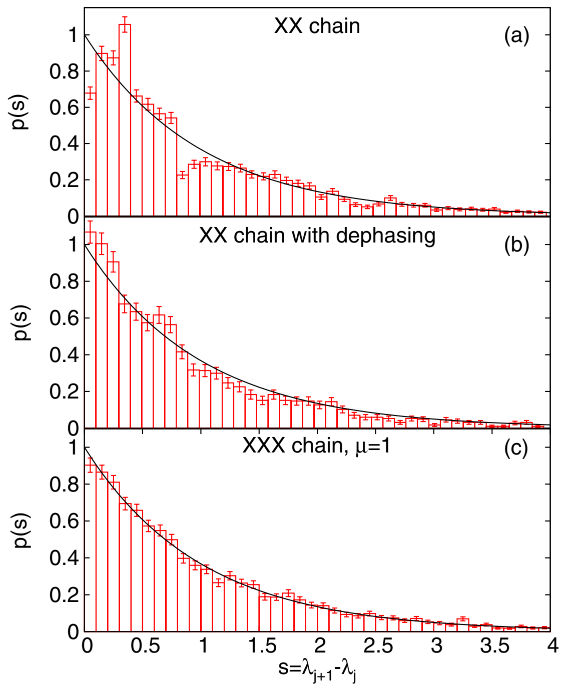

FIG. 1 (color online). LSD for NESSs of solvable nonequilibrium systems. (a) XX chain ($n = 16$, $Z = 10$). (b) XX chain with dephasing of strength $\gamma = 1$ ($n = 14$, $Z = 7$). (c) XXX chain with maximal driving $\mu = 1$ ($n = 20$, $Z = 5$, $\Delta = 1$). Cases (a) and (b) are for $\Gamma = 1$, $\mu = 0.2$, $\bar{\mu} = 0.3$, while (c) is for $\Gamma = 0.1$, $\mu = 1$, $\bar{\mu} = 0$.

that the LSD agrees, within statistical fluctuations, with the Poissonian statistics.

The last and least trivial solvable nonequilibrium case is the XXZ model ($b_j = 0$) at maximal driving $\mu = 1$, $\bar{\mu} = 0$, where ρ_0 can be written in terms of an infinite rank matrix product ansatz [24]. As one can see in Fig. 1(c) the LSD is again Poissonian which suggests the existence of Bethe equations for $\{\lambda_j\}$ and the Bethe ansatz form of eigenvectors of the NESS ρ_0 [34].

Nonsolvable open spin chains.—Here we consider two instances without dephasing: the XXZ model without a magnetic field $b_j = 0$ (solvable via the Bethe ansatz in its closed-system formulation [30] that has, however, so far evaded all attempts of finding a nonequilibrium solution at nonmaximal $\mu < 1$), and the XXZ chain in a staggered magnetic field for which the Hamiltonian is quantum chaotic.

For the XXZ model without a magnetic field ($b_j = 0$, $\gamma = 0$) we break parity P by using $\bar{\mu} = 0.3$ (antiunitary T is broken by Δ). For nonzero and nonmaximal driving μ, nonequilibrium exact solutions are not known. The LSD of the NESS, shown in Fig. 2(a) for $\Delta = 0.5$ and $\mu = 0.2$, agrees with the LSD of the GUE, describing complex quantum systems without antiunitary symmetry. This seems to indicate that such a nonequilibrium system is not solvable. Note that in nonequilibrium states that carry

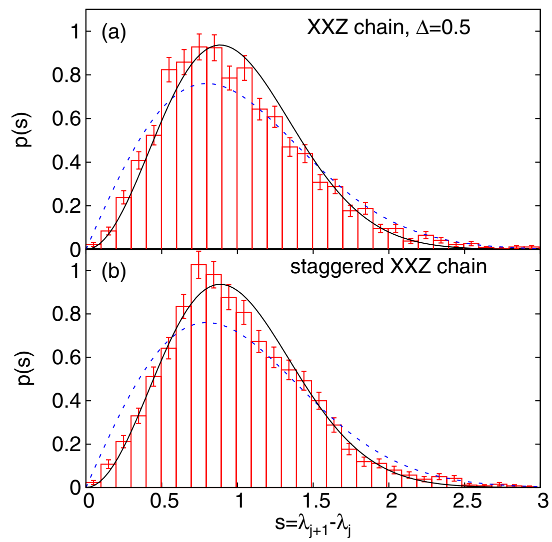

FIG. 2 (color online). LSD for NESSs of nonsolvable systems. (a) XXZ chain with $\Delta = 0.5$ ($\Gamma = 1$, $\mu = 0.2$, $\bar{\mu} = 0.3$). (b) XXZ chain with $\Delta = 0.5$ in a staggered field ($\mu = 0.1$, $\bar{\mu} = 0$, $\Gamma = 1$). Both cases are for $n = 14$ in the sector with $Z = 7$. Full black curve is the Wigner surmise for the GUE; the dotted blue curve is for the Gaussian orthogonal ensemble.

a current, as is the case in all NESSs studied here, we expect the LSD to display GUE statistics and not the one for the Gaussian orthogonal ensemble (GOE) irrespective of the symmetry class to which the Hamiltonian belongs (boundary driving will in general break the time-reversal symmetry of H). Fixing the system size n and increasing the anisotropy Δ (see Fig. 3), the LSD becomes increasingly Poisson-like. This might suggest that by increasing Δ the nonequilibrium XXZ model is perhaps amenable to an exact solution. Note that, not surprisingly, the limits $n \to \infty$ and $\Delta \to \infty$ do not commute. Taking a fixed Δ and increasing n one gets a GUE statistics while fixing n and increasing Δ one gets a Poissonian statistics.

Switching on an inhomogeneous magnetic field the XXZ model becomes nonintegrable even without driving. We use

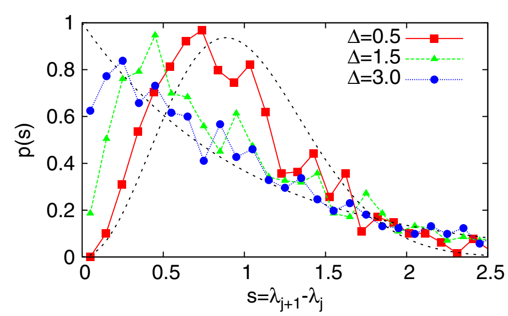

FIG. 3 (color online). LSD for NESSs of XXZ chain at different Δ. Increasing Δ at fixed size n, LSD becomes increasingly Poisson-like. All for $n = 13$ and the sector with $Z = 7$, $\mu = 0.2$, $\bar{\mu} = 0.3$, $\Gamma = 1$. Dotted black curves are Poissonian and GUE statistics.

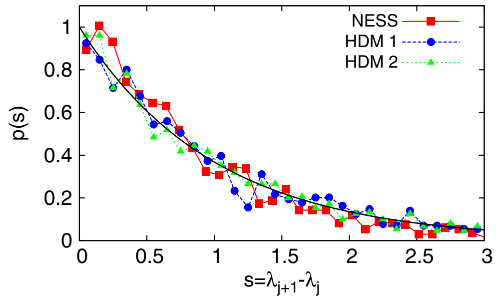

FIG. 4 (color online). LSD for two HDMs with the largest real eigenvalues $\Lambda_1 = -0.1019$ (blue), $\Lambda_2 = -0.2110$ (green), in the XX chain with dephasing $\gamma = 1$ ($\mu = 0.2$, $\bar{\mu} = 0.3$, $\Gamma = 1$, $n = 13$, $Z = 7$). Red squares indicate LSD for the NESS, $\Lambda_0 = 0$ [similar data, but different n, as in Fig. 1(b)].

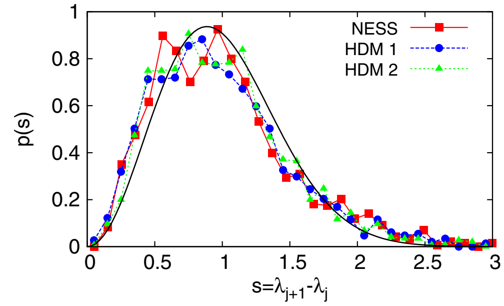

FIG. 5 (color online). LSD for two HDMs with the largest real eigenvalues $\Lambda_1 = -0.0943$ (blue), $\Lambda_2 = -0.1574$ (green), for the XXZ chain in a staggered field. Red points are for the NESS, $\Lambda_0 = 0$ ($\Delta = 0.5$, $\mu = 0.1$, $\bar{\mu} = 0$, $\Gamma = 1$, $n = 13$, $Z = 7$). Full black curve is the Wigner surmise for the GUE.

a period 3 staggered field $b_{3j} = 0$, $b_{3j+1} = -1$, $b_{3j+2} = -1/2$, and $\Delta = 0.5$, for which the Hamiltonian is quantum chaotic [31]. Out of equilibrium at $\mu = 0.1$ the NESS spectrum displays nice GUE statistics [see Fig. 2(b)].

Level spacing distribution of decay modes.—Apart from quadratic nonequilibrium models solvable by canonical quantization in Liouville space [16] no analytic solution for the decay modes of the quantum Liouvillian is known (in the limit $n \to \infty$). Calculating two nondegenerate HDMs with the longest decay time for the XX chain with dephasing, we obtain the LSD shown in Fig. 4. We can see that the statistics is for HDMs the same as for the NESS. This suggests that the decay modes in the XX chain with dephasing should also be amenable to an analytic calculation. For the nonsolvable XXZ chain in a staggered field the LSD in Fig. 5 agrees with the distribution for the GUE [HDMs for the XXZ chain without an external field (data not shown) also agree with the GUE]. Small deviations from the GUE theory visible in Fig. 5 are due to a smaller size $n = 13$ than $n = 14$ used in Fig. 2. We can mention that we also calculated LSD for HDMs in a maximally driven XXZ chain ($\mu = 1$). Unfortunately, the sizes available ($n \leq 13$) do not allow us to make a reliable conclusion about the behavior in the thermodynamic limit and hence to speculate about the exact solvability of decay modes. LSD data for HDMs at $n = 13$ (not shown) exhibit a Poissonian tail while at the same time showing also some level repulsion for small spacings.

Conclusion.—By analyzing Markovian master equations for a variety of boundary driven quantum spin chains we have put forward a generalization of the quantum chaos conjecture for nonequilibrium density operators. We show firm empirical evidence for the correspondence between exact solvability (integrability) and Poissonian level spacing statistics, on the one hand, and between nonintegrability and random-matrix statistics, on the other hand. Consistent results have been found inspecting also other spectral statistics, such as number variance (not shown). We identify possible new instances of solvable nonequilibrium steady states and decay modes.

Eigenvalues of many-body density operators can be interpreted in terms of occupation probabilities. Statistical fluctuations of these probabilities, discussed here, will influence information-theoretic quantities, like von Neumann entropy, and therefore are of general interest in nonequilibrium quantum physics.

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[1] M. L. Mehta, *Random Matrices* (Elsevier, New York, 2004), 3rd ed.
[2] E. V. Shurjak and J. J. M. Verbaarschot, Nucl. Phys. **A560**, 306 (1993).
[3] V. Plerou, P. Gopikrishnan, B. Rosenow, L. A. Nunes Amaral, T. Guhr, and H. E. Stanley, Phys. Rev. E **65**, 066126 (2002).
[4] M. V. Berry, Ann. Phys. (N.Y.) **131**, 163 (1981).
[5] G. Casati, F. Valz-Griz, and I. Guarneri, Lett. Nuovo Cimento **28**, 279 (1980).
[6] O. Bohigas, M.-J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[7] S. Müller, S. Heusler, P. Braun, F. Haake, and A. Altland, Phys. Rev. Lett. **93**, 014103 (2004).
[8] M. V. Berry and M. Tabor, Proc. R. Soc. A **356**, 375 (1977).
[9] D. Poilblanc, T. Ziman, J. Bellissard, F. Mila, and G. Montambaux, Europhys. Lett. **22**, 537 (1993).
[10] H.-J. Stöckmann, *Quantum Chaos: An Introduction* (Cambridge University Press, Cambridge, England, 1999).
[11] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. (N.Y.) **17**, 821 (1976); G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).
[12] Since the superoperator $\hat{\mathcal{L}}$ is hermiticity preserving, i.e., $\hat{\mathcal{L}}(\sigma^\dagger) = (\hat{\mathcal{L}}\sigma)^\dagger$ it follows that $\rho_m = (\rho_m)^\dagger$ whenever $\Lambda_m \in \mathbb{R}$ and nondegenerate.
[13] For reduced density operators describing subsystems of closed systems, RMT behavior has been found [14] irrespective of integrability of the total Hamiltonian.
[14] M. Mierzejewski, T. Prosen, D. Crivelli, and P. Prelovšek, Phys. Rev. Lett. **110**, 200602 (2013).

[15] R. A. Blythe and M. R. Evans, J. Phys. A **40**, R333 (2007).
[16] T. Prosen, New J. Phys. **10**, 043026 (2008).
[17] S. R. Clark, J. Prior, M. J. Hartmann, D. Jaksch, and M. B. Plenio, New J. Phys. **12**, 025005 (2010).
[18] D. Karevski and T. Platini, Phys. Rev. Lett. **102**, 207207 (2009).
[19] T. Prosen and T. H. Seligman, J. Phys. A **43**, 392004 (2010).
[20] K. Temme, M. M. Wolf, and F. Verstraete, New J. Phys. **14**, 075004 (2012).
[21] B. Horstmann, J. I. Cirac, and G. Giedke, Phys. Rev. A **87**, 012108 (2013).
[22] M. Žnidarič, J. Stat. Mech. (2010) L05002; M. Žnidarič, Phys. Rev. E **83**, 011108 (2011).
[23] V. Eisler, J. Stat. Mech. (2011) P06007.
[24] T. Prosen, Phys. Rev. Lett. **106**, 217206 (2011); **107**, 137201 (2011); D. Karevski, V. Popkov, and G. M. Schütz, Phys. Rev. Lett. **110**, 047201 (2013).
[25] T. Prosen and M. Žnidarič, J. Stat. Mech. (2009) P02035.
[26] R. Steinigeweg and J. Gemmer, Phys. Rev. B **80**, 184402 (2009); R. Steinigeweg, Phys. Rev. E **84**, 011136 (2011).
[27] M. Žnidarič, Phys. Rev. Lett. **106**, 220601 (2011).
[28] Note that a standard procedure of deriving the Lindblad master equation [29] results in Lindblad operators that are in general spatially nonlocal even if the system-bath interaction is local. However, local Lindbald operators result from an additional assumption that locally created excitations are not transferred to other sites within the time scale of the bath fluctuations. Another argument for using local Lindblad operators is simplicity and a higher likelihood of encountering exactly solvable cases.
[29] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University, New York, 2002).
[30] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, England, 1993).
[31] M. Žnidarič, T. Prosen, G. Benenti, G. Casati, and D. Rossini, Phys. Rev. E **81**, 051135 (2010).
[32] We chop off 10%–20% of the eigenvalues at the two ends of the spectrum and perform an unfolding $\lambda_j \to \mathcal{N}(\lambda_j)$ where $\mathcal{N}(\lambda)$ is a low-order polynomial fit to a level counting function, i.e., a number of eigenvalues less than λ.
[33] M. Žnidarič, J. Phys. A **43**, 415004 (2010).
[34] T. Prosen, E. Ilievski, and V. Popkov, New J. Phys. **15**, 073051 (2013).