Families of Quasilocal Conservation Laws and Quantum Spin Transport

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(Received 19 June 2013; published 2 August 2013)

For fundamental integrable quantum chains with deformed symmetries we outline a general procedure for defining a continuous family of quasilocal operators whose time derivative is supported near the two boundary sites only. The program is implemented for a spin $1/2$ XXZ chain, resulting in improved rigorous estimates for the high temperature spin Drude weight.

DOI: 10.1103/PhysRevLett.111.057203 PACS numbers: 75.10.Pq, 02.30.Ik, 03.65.Yz, 05.60.Gg

Introduction.—The quantum dynamics of locally interacting systems in one dimension [1] continues to pose fundamental challenges to theorists. For example, complete integrability [2], which implies the existence of a macroscopic number (i.e. of the same order as the number of particles or degrees of freedom) of local conservation laws can prevent a system from thermalizaing [3,4] or stop a current from decaying [5–7], but precise conditions for when and mechanisms for how this can happen are still generally unclear. For instance, it is possible that due to general symmetry arguments all the local conserved quantities following from the quantum inverse scattering method [2] are irrelevant for the interesting physical observables under study, like magnetization or spin current. This happens, for example, in the anisotropic Heisenberg spin $1/2$ chain, the so-called XXZ model, and allows for a surprising suggestion [8,9] of spin diffusion in the Ising-like regime of high spin-coupling anisotropy $|\Delta| > 1$. Nevertheless, for $|\Delta| \le 1$, numerical [10,11] and experimental [12,13] evidence exists for anomalous, or even ballistic transport, which is characterized by positivity of spin Drude weight, despite the fact that at zero magnetization (or in the absence of an external magnetic field) the spin current is orthogonal to all local conserved operators [14] and so the Mazur bound [15] for the Drude weight vanishes [5].

The problem has recently been resolved by finding a "missing conservation law" Z [16]; namely, it has been shown that a quasilocal operator exists in the form of a rapidly converging series of local operators that almost commutes with the Hamiltonian in the sense that the residual terms are supported only near the boundary of the chain and Drude weight can still be bounded away from zero rigorously [17]. However, this exotic new object Z appeared rather mysterious as the technique for deriving it [16] did not seem connected to integrability structures such as transfer operators and Yang-Baxter equations (YBE); neither did it gave any clue on how it may be generalized to other models. In this Letter we answer these puzzles by deriving a whole family of quasilocal conservation laws as a function of a complex parameter. These new objects, including Z of Ref. [16] as a special case, are derived from a novel, so-called *highest-weight* quantum Yang-Baxter transfer operator based on an infinitely dimensional complex spin s representation of the quantum group $U_q(\mathfrak{sl}_2)$, which is the symmetry of the XXZ model. Quasilocality here emerges as a consequence of differentiation with respect to s at the trivial point $s = 0$, unlike in the standard case [2,14] where locality is a consequence of taking the logarithm of the *trace* transfer operator in the fundamental representation $s = 1/2$. We show how the new continuous family of conservation laws can be applied to yield improved rigorous Mazur bound [17] on spin Drude weight. We focus our analysis to the case of a XXZ spin $1/2$ chain, but it should be generalizable to other fundamental integrable models sharing quantum group Yang-Baxter structure.

Holomorphic family of almost-conserved quasilocal operators.—The starting point is in acknowledging (see e.g., [18–20]) a general Yang-Baxter equation in a triple vector space $\mathcal{V}_{s_1} \otimes \mathcal{V}_{s_2} \otimes \mathcal{V}_{s_3}$ where $s_1, s_2, s_3 \in \mathbb{C}$ denote arbitrary complex representation parameters for generally infinitely dimensional representations of quantum group $U_q[\mathfrak{sl}(2)]$. These so-called Verma modules $\mathcal{V}_s$, spanned by a semi-infinite orthonormal basis $|k\rangle$, $k \in \mathbb{Z}^+$, are generated by deformed spin-s operators

$$\mathbf{S}_s^z = \sum_{k=0}^{\infty} (s-k)|k\rangle\langle k|,$$
$$\mathbf{S}_s^+ = \sum_{k=0}^{\infty} \frac{\sin(k+1)\lambda}{\sin\lambda} |k\rangle\langle k+1|, \tag{1}$$
$$\mathbf{S}_s^- = \sum_{k=0}^{\infty} \frac{\sin(2s-k)\lambda}{\sin\lambda} |k+1\rangle\langle k|,$$

satisfying the quantum group relations $[\mathbf{S}_s^+, \mathbf{S}_s^-] = \sin[2\lambda \mathbf{S}_s^z]/\sin\lambda$, $[\mathbf{S}_s^z, \mathbf{S}_s^\pm] = \pm\mathbf{S}_s^\pm$. For clarity of notation we use bold symbols to denote objects that are not scalars over infinitely dimensional module $\mathcal{V}_s$. For $s \in \mathbb{Z}^+/2$, $\mathcal{V}_s$ is reducible to a finite, $2s+1$ dimensional irreducible representation. The deformation parameter $q = \exp(i\lambda)$ is related to the anisotropy parameter $\Delta = \cos\lambda$ of the n-spin $1/2$ XXZ Heisenberg chain with Hamiltonian

$$H_n = \sum_{x=1}^{n-1} \mathbb{1}_{2^{x-1}} \otimes h \otimes \mathbb{1}_{2^{n-x-1}},$$
$$h = 2\sigma^+ \otimes \sigma^- + 2\sigma^- \otimes \sigma^+ + \Delta\sigma^z \otimes \sigma^z, \tag{2}$$

which can be considered as an operator over $\mathcal{V}_{1/2}^{\otimes n} \simeq \mathbb{C}^{2^n}$ and $\sigma^{\pm,z}$, $\sigma^0 = \mathbb{1}_2$ is a set of standard Pauli matrices.

Let us define a two-parametric Lax operator in terms of the universal R matrix over $\mathcal{V}_s \otimes \mathcal{V}_{1/2}$ [19], i.e. a 2×2 matrix with entries in End$(\mathcal{V}_s)$

$$\mathbf{L}(\varphi, s) = \begin{pmatrix} \sin(\varphi + \lambda \mathbf{S}_s^z) & (\sin\lambda)\mathbf{S}_s^- \\ (\sin\lambda)\mathbf{S}_s^+ & \sin(\varphi - \lambda\mathbf{S}_s^z) \end{pmatrix}. \tag{3}$$

Then the YBE in $\mathcal{V}_s \otimes \mathcal{V}_{s'} \otimes \mathcal{V}_{1/2}$ together with the fact that $\langle 0| \otimes \langle 0|$ ($|0\rangle \otimes |0\rangle$) is a left (right) eigenvector of the R matrix [21] over $\mathcal{V}_s \otimes \mathcal{V}_{s'}$ guarantees commutativity of the highest weight transfer operators [22]

$$W_n(\varphi, s) = \langle 0|\mathbf{L}(\varphi, s)^{\otimes n}|0\rangle. \tag{4}$$

Namely, for any pair of spectral parameters $\varphi, \varphi' \in \mathbb{C}$ and representation parameters $s, s' \in \mathbb{C}$, we have

$$[W_n(\varphi, s), W_n(\varphi', s')] = 0. \tag{5}$$

Note that the special case $S = W_n(\pi/2, s)$ is exactly the Cholesky factor of the nonequilibrium steady state density operator $SS^\dagger$ [23,24] of the boundary driven XXZ chain, with Lindblad jump operators $L_1 = \sqrt{\varepsilon}\sigma^- \otimes \mathbb{1}_{2^{n-1}}$, $L_2 = \sqrt{\varepsilon}\mathbb{1}_{2^{n-1}} \otimes \sigma^+$, if $\cot(s\lambda) = \varepsilon/(2i\sin\lambda)$.

The operators $W_n(\varphi, s)$ are in general nonlocal and are not commuting with the Hamiltonian H_n; however, setting spectral parameters to $\varphi' = \varphi + \delta$ in the YBE and expanding to first order in δ results in a fundamental divergence relation for local two-site commutators [25,26]

$$[h, \mathbf{L} \otimes \mathbf{L}] = 2\sin\lambda(\mathbf{L} \otimes \mathbf{L}_\varphi - \mathbf{L}_\varphi \otimes \mathbf{L}), \tag{6}$$

where $\mathbf{L} \equiv \mathbf{L}(\varphi, s)$, $\mathbf{L}_\varphi \equiv \partial_\varphi \mathbf{L}(\varphi, s) = \cos\varphi\cos(\lambda\mathbf{S}_s^z)\otimes\sigma^0 - \sin\varphi\sin(\lambda\mathbf{S}_s^z)\otimes\sigma^z$.

Of fundamental importance to gain quasilocality in these objects is a derivation with respect to a complex (deformed spin) representation parameter at $s = 0$, which is implied by the following observation.

Lemma.—Let us define a modified auxiliary space $\tilde{\mathcal{V}}$ with a split vacuum, namely, $|0\rangle$ replaced by a pair of distinct highest weight states $|\mathrm{L}\rangle$ and $|\mathrm{R}\rangle$, i.e. $\tilde{\mathcal{V}}$ being spanned by a formal orthonormal basis $\{|\mathrm{L}\rangle, |\mathrm{R}\rangle, |1\rangle, |2\rangle, \ldots\}$. Let $\tilde{\mathbf{S}}^\alpha = \mathbf{S}_0^\alpha|_{\tilde{\mathcal{V}}}$ denote projected spin operators, which are essentially given by (1) with summation index k running from 1 [27], and define a modified Lax matrix

$$\tilde{\mathbf{L}}(\varphi) = \sum_{\alpha\in\{0,\pm,z\}} \tilde{\mathbf{L}}^\alpha(\varphi) \otimes \sigma^\alpha \tag{7}$$

with components

$$\tilde{\mathbf{L}}^0(\varphi) = |\mathrm{L}\rangle\langle\mathrm{L}| + |\mathrm{R}\rangle\langle\mathrm{R}| + \cos(\lambda\tilde{\mathbf{S}}^z),$$
$$\tilde{\mathbf{L}}^z(\varphi) = \cot\varphi\sin(\lambda\tilde{\mathbf{S}}^z),$$
$$\tilde{\mathbf{L}}^+(\varphi) = |1\rangle\langle\mathrm{R}| + \frac{\sin\lambda}{\sin\varphi}\tilde{\mathbf{S}}^-,$$
$$\tilde{\mathbf{L}}^-(\varphi) = |\mathrm{L}\rangle\langle 1| + \frac{\sin\lambda}{\sin\varphi}\tilde{\mathbf{S}}^+.$$

Consequently, we define also the corresponding modified highest weight transfer operator

$$Z_n(\varphi) = \langle\mathrm{L}|\tilde{\mathbf{L}}(\varphi)^{\otimes n}|\mathrm{R}\rangle. \tag{8}$$

Then, the normalized s derivative at $s = 0$ can be expressed as

$$\frac{1}{(\sin\varphi)^n}\partial_s W_n(\varphi, s)|_{s=0} = \frac{2\lambda}{\sin\lambda} Z_n(\varphi) + \lambda\cot\varphi M_n^z, \tag{9}$$

where $M_n^z = \sum_{x=1}^n \mathbb{1}_{2^{x-1}} \otimes \sigma^z \otimes \mathbb{1}_{2^{n-x}}$ is the total magnetization operator.

The proof is just a formal expression of the fact that at $s = 0$ the transitions $|0\rangle \to |0\rangle$ in $\partial_s\langle 0|\mathbf{L}(\varphi, s)^{\otimes n}|0\rangle|_{s=0}$, expressed via the Leibniz rule applied over an n-fold matrix product operator, are only possible (i) via virtual states $|1\rangle, |2\rangle, \ldots$ if the s derivative "acts" on the amplitude $\langle 1|\mathbf{S}_s^-|0\rangle$ (1), which otherwise would vanish as $s = 0$, or (ii) directly where the s derivative acts on the amplitude at $\langle 0|\mathbf{S}_s^z|0\rangle$. The cases (i) and (ii) correspond to the first and second term on the right-hand side of Eq. (9), respectively. Note that all the operators under discussion commute with M_n^z, $[W_n(\varphi, s), M_n^z] = [H_n, M_n^z] = 0$; hence also $Z_n(\varphi)$ form a commuting family

$$[Z_n(\varphi), Z_n(\varphi')] = 0, \quad \forall\ \varphi, \varphi' \in \mathbb{C}. \tag{10}$$

Equation (8) generates a matrix product operator

$$Z_n = \sum_{\alpha_1,\alpha_2\ldots,\alpha_n} \langle\mathrm{L}|\tilde{\mathbf{L}}^{\alpha_1}\tilde{\mathbf{L}}^{\alpha_2}\cdots\tilde{\mathbf{L}}^{\alpha_n}|\mathrm{R}\rangle\sigma^{\alpha_1}\otimes\sigma^{\alpha_2}\otimes\cdots\sigma^{\alpha_n}, \tag{11}$$

which can be written as a sum of local terms, since $\langle\mathrm{L}|\tilde{\mathbf{L}}^\alpha = \delta_{\alpha,0}\langle\mathrm{L}| + \delta_{\alpha,-}\langle 1|$, $\tilde{\mathbf{L}}^\alpha|\mathrm{R}\rangle = \delta_{\alpha,0}|\mathrm{R}\rangle + \delta_{\alpha,+}|1\rangle$,

$$Z_n = \sum_{r=2}^{n}\sum_{x=0}^{n-r} \mathbb{1}_{2^x} \otimes q_r \otimes \mathbb{1}_{2^{n-r-x}}, \tag{12}$$

where q_r is an r-site density, i.e. an element of End$(\mathcal{V}_{1/2}^{\otimes r})$ which acts nontrivially on sites 1 and r. In other words, q_r is of exact matrix product operator form (11) for $n = r$ with $\alpha_1 = -$, $\alpha_r = +$. We define [17] the operator $Z \equiv Z_{n=\infty}$ of the infinite chain to be quasilocal if $\exists\gamma$, $\xi > 0$ such that $\|q_r\| < \gamma\exp(-\xi r)$, and we call the operator sequence Z_n to be almost conserved if for any n, $[H_n, Z_n] = \sum_{r=1}^n (b_r \otimes \mathbb{1}_{2^{n-r}} - \mathbb{1}_{2^{n-r}} \otimes b_r)$, where $b_r \in$ End$(\mathcal{V}_{1/2}^{\otimes r})$ and $\exists\gamma', \xi' > 0$ such that $\|b_r\| < \gamma'\exp(-\xi' r)$. We have shown in previous work [16,17] that existence of quasilocal almost-conserved operators (QLAC) implies nontrivial bounds on ballistic transport.

We are now in position to state the main result.

Theorem.—For a dense set of commensurate easy-plane anisotropies $\lambda = \pi l/m$, $l, m \in \mathbb{Z}^+$, $l \le m > 0$, the operators $Z(\varphi)$ are strictly quasilocal and almost conserved for all $\varphi \in \mathcal{D}_m \subset \mathbb{C}$ where $\mathcal{D}_m = \{\varphi; |\mathrm{Re}\varphi - (\pi/2)| < (\pi/2m)\}$ is an open vertical strip of width π/m centered around $\varphi_0 = \pi/2$. Furthermore, $Z(\varphi)$ is holomorphic on $\mathcal{D}_m$.

Proof.—We start by tensor multiplying the local divergence relation (6) by $\mathbf{L}^{\otimes(j-1)} \otimes \bullet \otimes \mathbf{L}^{\otimes(n-j-1)}$, then summing over $j = 1 \ldots n-1$, taking the highest weight state expectation value $\langle 0| \bullet |0\rangle$, and finally differentiating $\partial_s \bullet |_{s=0}$. Using the Lemma (9) and carefully bookkeeping all the terms, we finally arrive at the key identity

$$[H_n, Z_n(\varphi)] = \sigma^z \otimes \mathbb{1}_{2^{n-1}} - \mathbb{1}_{2^{n-1}} \otimes \sigma^z - 2\sin\lambda\cot\varphi(\sigma^0 \otimes Z_{n-1}(\varphi) - Z_{n-1}(\varphi) \otimes \sigma^0). \tag{13}$$

Exploring a one-element-per-row property of the matrices $\tilde{\mathbf{L}}^\alpha$, the Hilbert-Schmidt product $(A, B) := 2^{-n}\mathrm{tr}(A^\dagger B)$ of any pair of $Z_n(\varphi)$ can be calculated in terms of a two-parametric transfer matrix [28]

$$K_n(\varphi, \varphi') := (Z_n(\bar{\varphi}), Z_n(\varphi')) = \langle \mathrm{L}|\mathbf{T}(\varphi, \varphi')^n|\mathrm{R}\rangle, \tag{14}$$

$$\mathbf{T}(\varphi, \varphi') := |\mathrm{L}\rangle\langle\mathrm{L}| + |\mathrm{R}\rangle\langle\mathrm{R}| + \frac{|\mathrm{L}\rangle\langle 1|}{2} + \frac{|1\rangle\langle\mathrm{R}|}{2} + \mathbf{T}'(\varphi, \varphi'),$$

$$\mathbf{T}'(\varphi, \varphi') := \sum_{k=1}^{\infty}\bigg\{(\cos^2(k\lambda) + \cot\varphi\cot\varphi'\sin^2(k\lambda))|k\rangle\langle k| + \frac{|\sin(k\lambda)\sin(k+1)\lambda|}{2\sin\varphi\sin\varphi'}(|k\rangle\langle k+1| + |k+1\rangle\langle k|)\bigg\}. \tag{15}$$

For $\lambda = \pi l/m$, the transition $|m\rangle \leftarrow |m-1\rangle$ is forbidden, $\langle m|\mathbf{T}|m-1\rangle = 0$, so $\mathbf{T}$ can be replaced by a $(m+1) \times (m+1)$ matrix truncated to a finite set of states $|\mathrm{L}\rangle, |\mathrm{R}\rangle, |1\rangle, \ldots, |m-1\rangle$ with a symmetric tridiagonal matrix $\mathbf{T}'$ being its orthogonal projection to the last $m-1$ states. Then we prove a sequence of statements. (i) $\mathbf{T}'$ is strictly contracting, i.e. for all its eigenvalues τ_j, $j = 1 \ldots, m-1$, sorted as $|\tau_1| > |\tau_2| > \ldots$, we have $|\tau_j| < 1$ if $\varphi, \varphi' \in \mathcal{D}_m$. First, let us assume $\varphi' = \bar{\varphi}$ and write $\mathrm{Re}\varphi = (\pi/2) + u$. Defining a positive diagonal matrix $\mathbf{D} = \sum_{k=1}^{m-1}|\sin(\pi lk/m)||k\rangle\langle k|$, and tridiagonal Toeplitz matrix $\mathbf{A} = \sum_{k=1}^{m-1}\cos(2u)|k\rangle\langle k| - (1/2) \times \sum_{k=1}^{m-2}(|k\rangle\langle k+1| + |k+1\rangle\langle k|)$, we have $\mathbb{1} - \mathbf{T}' = |\sin\varphi|^{-2}\mathbf{DAD}$. Matrix elements of $\mathbf{T}'$ are real and non-negative so leading eigenvalue should be positive $\tau_1 > 0$, and $\mathbf{T}'$ is contracting if $\mathbb{1} - \mathbf{T}' > 0$. This is equivalent to condition $\mathbf{A} > 0$ which holds if $|u| < \pi/2m$, i.e. $\varphi \in \mathcal{D}_m$. For general $\varphi, \varphi' \in \mathcal{D}_m$, $\mathbf{T}'$ is still contracting as a consequence of Cauchy-Schwartz inequality $|K_n(\varphi, \varphi')|^2 \le K_n(\bar{\varphi}, \varphi)K_n(\bar{\varphi}', \varphi')$. (ii) τ_j are also eigenvalues of $\mathbf{T}$, whereas the eigenvectors $|\tau_j\rangle'$ of $\mathbf{T}'$ map to the corresponding eigenvectors of $\mathbf{T}$ via $|\tau_j\rangle = |\tau_j\rangle' + |\mathrm{L}\rangle\langle 1|\tau_j\rangle'/(2\tau_j - 2)$. (iii) Furthermore, $\mathbf{T}$ has an eigenvalue $\tau_0 = 1$ of multiplicity 2 with a single eigenvector $|\tau_0\rangle = |\mathrm{L}\rangle$ and a defective eigenvector $|\psi\rangle = \psi_\mathrm{R}|\mathrm{R}\rangle + \sum_j \psi_j|j\rangle$, $(\mathbf{T} - \mathbb{1})|\psi\rangle = |\tau_0\rangle$ whose components can be calculated from bottom-up substitution using explicit form (14), resulting in a recurrence $\psi_{m-k} = [T_{m-k,m-k-1}/(1 - T_{m-k,m-k})]C_{k-1}^{-1}\psi_{m-k-1}$ and $\psi_1 = 2$, $\psi_\mathrm{R} = 2[T_{21}/(1 - T_{22})]C_{m-2}^{-1}$ where C_k form a continued fraction sequence $C_0 = 1$, $C_{k+1} = 1 - 1/[4\cos^2(\varphi + \varphi')C_k]$. Implementing Jordan decomposition of Ref. [16] one finally obtains an explicit estimation $K_n(\varphi, \varphi') = nK(\varphi, \varphi') + \mathcal{O}(\tau_1^n)$ where

$$K(\varphi, \varphi') = -\frac{\sin\varphi\sin\varphi'}{2\sin^2(\pi l/m)}\frac{\sin[(m-1)(\varphi + \varphi')]}{\sin[m(\varphi + \varphi')]}. \tag{16}$$

Note that $K(\varphi, \varphi')$ is nonsingular when $\varphi, \varphi' \in \mathcal{D}_m$, whereas $K(\bar{\varphi}, \varphi) = \lim_{n\to\infty}(Z_n(\varphi), Z_n(\varphi))/n$ is becoming singular exactly for $\mathrm{Re}\varphi = \pi/2 \pm \pi/(2m)$, i.e. on $\partial\mathcal{D}_m$. For densities q_r (12) we write $(q_r(\varphi), q_r(\varphi)) = \langle 1|\mathbf{T}'(\bar{\varphi}, \varphi)^r|1\rangle$, following (15), implying together with elementary operator-norm inequality $\|A\|^2 \le (A, A)$:

$$\|q_r(\varphi)\| \le \gamma|\tau_1(\bar{\varphi}, \varphi)|^{r/2}, \quad \text{for some } \gamma > 0. \tag{17}$$

This proves quasilocality of $Z(\varphi)$ for $\varphi \in \mathcal{D}_m$ with exponent $\xi = -(1/2)\log|\tau_1| > 0$. Almost-conservation with the same exponent $\xi' = \xi$ follows by rewriting the second line of Eq. (13) as $Z_{n-1}(\varphi) \otimes \sigma^0 - \sigma^0 \otimes Z_{n-1}(\varphi) = \sum_{r=2}^{n}(q_r \otimes \mathbb{1}_{2^{n-r}} - \mathbb{1}_{2^{n-r}} \otimes q_r)$. $Z(\varphi)$ is also holomorphic on $\mathcal{D}_m$ as it is given in terms of an exponentially converging sum, in operator norm, of strictly local operators, each of which is holomorphic in φ. ■

We note that $Z_n(\pi/2)$ is exactly an isolated QLAC $Z^\dagger$ constructed in Ref. [16] via an alternative model-specific method, whereas the technique described here should be readily generalizable to other integrable models with deformed symmetries.

Integral form of Mazur bound for spin Drude weight.—Here we will show how the continuous family of QLAC $Z_n(\varphi)$ can be applied to rigorously estimate the spin Drude weight D which yields the ballistic contribution to the real part of spin conductivity $\sigma'(\omega) = 2\pi D\delta(\omega) + \sigma'_{\mathrm{reg}}(\omega)$. Within the linear response theory the Drude weight can be expressed in terms of a time-correlation function as $D = \lim_{t\to\infty}\lim_{n\to\infty}(\beta/2nt)\int_0^t dt'\langle J_n(t')J_n\rangle_\beta$, where $J_n(t) = e^{itH_n}J_n e^{-itH_n}$, $\langle\bullet\rangle_\beta = \mathrm{tr}(e^{-\beta H_n}\bullet)/\mathrm{tr}e^{-\beta H_n}$, and $J_n = \sum_{x=1}^{n-1}\mathbb{1}_{2^{x-1}} \otimes j \otimes \mathbb{1}_{2^{n-x-1}}$ is the spin current with density $j = i\sigma^+ \otimes \sigma^- - i\sigma^- \otimes \sigma^+$.

Limiting ourselves, for simplicity, to infinite temperature $\beta = 0$ we apply the rigorous form of the Mazur bound [5,15], namely, Theorem 2 of Ref. [17], stating that $D \ge \lim_{n\to\infty}(\beta/2n)\sum_{k,l}(J_n, Q_k)(U^{-1})_{k,l}(Q_l, J_n)$ where $U_{k,l} = (Q_k, Q_l)$ is a positive definite matrix and $\{Q_k\}$ is an arbitrary set of linearly independent QLACs (noting that they need not be Hermitian). Here we take an incountable

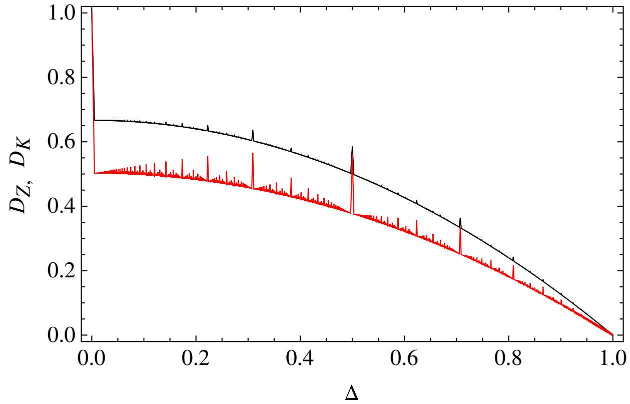

FIG. 1 (color online). Optimized Mazur bound D_K (20) (black) versus the bound D_Z of Ref. [16] (red or gray) which is based on a single quasilocal almost-conserved operator $Z_n(\pi/2)$.

continuum of them, namely, $\{Z_n(\varphi)\} \cup \{Z_n^\dagger(\varphi)\}$ labelled by points φ from a two-dimensional analyticity strip $\varphi \in \mathcal{D}_m$. Using elementary identities $(J_n, Z_n(\varphi)) = -(J_n, Z_n^\dagger(\varphi)) \equiv -i(n-1)/4$, and $(Z_n(\varphi), Z_n^\dagger(\varphi')) \equiv 0$, we arrive at the Drude weight estimate $D \geq (\beta/4)D_K$ with

$$D_K = \frac{1}{4}\int_{\mathcal{D}_m} d^2\varphi f(\varphi), \tag{18}$$

where $f(\varphi)$ is the solution of the complex-plane Fredholm integral equation of the first kind

$$\int_{\mathcal{D}_m} d^2\varphi' K(\varphi, \varphi') f(\varphi') = 1, \quad \varphi \in \mathcal{D}_m. \tag{19}$$

The kernel $K(\varphi, \varphi')$ defines a positive definite operator, substituting for the matrix $(1/n)U_{k,l}$ in [17], which we essentially have to invert. Fortunately, the form of solution can be guessed in our case (16), namely, $f(\varphi) = c/|\sin\varphi|^4$ where c is a constant that can be determined by elementary integration, yielding an explicit, closed-form expression for the Drude weight bound (see also Fig. 1)

$$D_K = \frac{\sin^2(\pi l/m)}{\sin^2(\pi/m)}\left(1 - \frac{m}{2\pi}\sin\left(\frac{2\pi}{m}\right)\right). \tag{20}$$

This is a nontrivial improvement over the previous lower bound $D_Z = [m/(2(m-1))]\sin^2(\pi l/m) = [m/(2(m-1))] \times (1-\Delta^2)$ [16] based on a single QLAC $Z_n(\pi/2)$, $D_K > D_Z$, but again is a nowhere differentiable function of Δ and, remarkably, agrees with one of the debatable Bethe ansatz results [29] at $\lambda = \pi/m$. It seems we have now fully explored the known Yang-Baxter structure of the problem; hence, we dare to conjecture that our bound (20) should in fact be saturated. One might suggest that higher s derivatives $(d/ds)^k W(\varphi, s)|_{s=0}$ could also be candidates for further independent QLACs; however, a brief inspection shows that already the second derivative $k=2$ at $\varphi = \pi/2$ is a nonlocal operator.

Conclusion.—We have outlined a procedure for derivation of families of quasilocal conservation laws of the XXZ chain that are orthogonal to previously known [14] strictly local conserved quantities. The latter are given, for periodic boundary conditions, in terms of logarithmic φ derivatives of a trace of a monodromy matrix in fundamental representation $F_n^{(k)} = (d/d\varphi)^k \log\mathrm{tr}\mathbf{L}(\varphi, 1/2)^{\otimes n}|_{\varphi=\lambda/2}$ and are irrelevant for the spin transport in the absence of external magnetic fields since $(J_n, F_n^{(k)}) = 0$. The former, however, can be derived using related though more involved integrability concepts, namely, in terms of derivation of a highest-weight (vacuum) diagonal element of a quantum monodromy matrix with respect to a complex spin representation parameter at $s = 0$.

We thank I. Affleck for pointing out how to construct defective eigenvectors of matrices of type (14) and acknowledge support by Slovenian ARRS Grant No. P1-0044.

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[27] Note that such truncated spin operators $\tilde{\mathbf{S}}^{\alpha}$ in fact generate $\mathcal{V}_{-1}$, while $\mathcal{V}_0$ is one dimensional.

[28] We note [16] that $\mathbf{T} := \sum_{\alpha}(1/2)\mathrm{tr}[(\sigma^{\alpha})^{\dagger}\sigma^{\alpha}]\tilde{\mathbf{L}}^{\alpha}(\varphi) \otimes \tilde{\mathbf{L}}^{\alpha}(\varphi')$ preserves the diagonal subspace, identifying $|k\rangle \otimes |k\rangle \rightarrow |k\rangle$. In addition, we apply diagonal similarity transformation that makes the transfer matrix $\mathbf{T}'$ (15) symmetric.

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