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Generic examples of $\mathbb{PT}$-symmetric qubit (spin-1/2) Liouvillian dynamics

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We outline two general classes of examples of $\mathbb{PT}$-symmetric quantum Liouvillian dynamics of open many-qubit systems, namely, interacting hard-core bosons (or more general XYZ-type spin-1/2 systems) having either (i) pure dephasing noise or (ii) solely single-particle (spin) injection (absorption) incoherent processes. The concept of $\mathbb{PT}$ symmetry is defined following Prosen [Phys. Rev. Lett. **109**, 090404 (2012)] as a formal quantum Liouville-space analog of the parity-time $\mathcal{PT}$ symmetry used in nonconservative classical systems with symmetrically distributed gain and loss.

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The concept of $\mathcal{PT}$ symmetry has been introduced [1] as a mathematical framework for studying non-Hermitian operators with spectra having a certain symmetry, for example, lying on the real line. In recent years, $\mathcal{PT}$ symmetry has been studied extensively, both theoretically [2–6] and experimentally in the context of optics [7,8] and electric circuits [9]. Very recently, a formally analogous concept has been proposed [10] on the level of master symmetries of quantum master equations [11–13] (Liouville equations) describing open quantum systems. It has been shown that the existence of quantum $\mathbb{PT}$ symmetry generically implies the existence of a spontaneous symmetry-breaking transition, such that for sufficiently weak coupling to the environment all coherences (off-diagonal matrix elements of the system's density matrix in the energy eigenbasis) decay with the same (uniform) damping rate.

The aim of this Brief Report is to provide two general classes of practically interesting examples for a recent construction of quantum Liouvillian $\mathbb{PT}$ symmetry [10]. We use notation exactly as introduced in Ref. [10] and refer to definitions stated there.

We start by proving a simple observation which seems to be crucial for a construction of general examples of $\mathbb{PT}$-symmetric Liouvillian systems:

Lemma. The Liouvillian flow is $\mathbb{PT}$ symmetric, if the following conditions are fulfilled:

(i) The parity superoperator can be represented as

$$\hat{\mathcal{P}}\rho = U\rho W,$$

where $U,W \in \mathcal{B}(\mathcal{H})$ are two unitary operators satisfying

$$U^2 = W^2 = \mathbb{1}$$

and

$$[H,U] = [H,W] = 0.$$

(ii) There exists an $M \times M$ real orthogonal reflection matrix $Z \in \mathrm{O}(M,\mathbb{R})$, satisfying $Z^2 = \mathbb{1}_M$, such that

$$UL_m = -\sum_{m'=1}^{M} Z_{m,m'} L_{m'}^\dagger U,$$

$$WL_m = \sum_{m'=1}^{M} Z_{m,m'} L_{m'}^\dagger W.$$

(iii) $\{L_m, L_m^\dagger\} = c_m \mathbb{1}$, for some $c_m \in \mathbb{R}$.

Proof. Using (iii), $\hat{\mathcal{D}}'$ can be written as

$$\hat{\mathcal{D}}'\rho = \sum_m \left(2L_m\rho L_m^\dagger - \frac{1}{2}\{[L_m^\dagger, L_m],\rho\}\right), \quad (1)$$

while in the Hilbert-Schmidt metric

$$(\hat{\mathcal{D}}')^\dagger\rho = \sum_m \left(2L_m^\dagger\rho L_m - \frac{1}{2}\{[L_m^\dagger, L_m],\rho\}\right). \quad (2)$$

Then, using (ii),

$$\begin{aligned}\hat{\mathcal{D}}'\hat{\mathcal{P}}\rho &= \hat{\mathcal{D}}'(U\rho W)\\ &= -U\sum_m \left(2L_m^\dagger\rho L_m - \frac{1}{2}\{[L_m^\dagger, L_m],\rho\}\right)W\\ &= -\hat{\mathcal{P}}(\hat{\mathcal{D}}')^\dagger\rho. \end{aligned} \quad (3)$$

Finally, using (i), we have also (writing $(\mathrm{ad}\, H)\rho = [H,\rho]$)

$$(i\,\mathrm{ad}\, H)\hat{\mathcal{P}} = \hat{\mathcal{P}}(i\,\mathrm{ad}\, H) = -\hat{\mathcal{P}}(i\,\mathrm{ad}\, H)^\dagger, \quad (4)$$

and therefore

$$\hat{\mathcal{L}}'\hat{\mathcal{P}} = -\hat{\mathcal{P}}(\hat{\mathcal{L}}')^\dagger, \quad (5)$$

i.e., we have the definition of $\mathbb{PT}$ symmetry [Eq. (8) of [10]]. ■

One quite restricted example of the application of $\mathbb{PT}$-symmetric quantum Liouvillian dynamics has been provided in Ref. [10]. Here we show that applications are in fact quite abundant and should not be difficult to realize in experimentally accessible situations.

Example 1. Consider n spins 1/2, or qubits, described by Pauli matrices $\sigma_j^{x,y,z,\pm}$, $j = 1,\ldots,n$. Take an arbitrary *two-spin* Hamiltonian with a *longitudinal* external field

$$H = \sum_{j<k}\left(J_{j,k}^x\sigma_j^x\sigma_k^x + J_{j,k}^y\sigma_j^y\sigma_k^y + J_{j,k}^z\sigma_j^z\sigma_k^z\right) + \sum_j h_j\sigma_j^x, \quad (6)$$

and up to $M = n$ local Hermitian "dephasing" Lindblad operators

$$L_j = \gamma_j\sigma_j^z, \quad j = 1,\ldots,n. \quad (7)$$

The interaction strengths $J_{j,k}^{x,y,z}$, magnetic field strengths h_j, and local dephasing rates γ_j can be completely arbitrary. The Liouvillian dynamics defined with respect to the Hamiltonian

H and Lindblad operators $\{L_j\}$ is $\mathbb{PT}$ symmetric, according to the Lemma above, if we take

$$U = \prod_{j=1}^{n} \sigma_j^x, \quad W = \mathbb{1}. \tag{8}$$

When $J_{j,k}^x \equiv J_{j,k}^y$, $h_j \equiv 0$, the model represents a general interacting system of hard-core bosons with zero (or constant) chemical potential. Then, the Lindblad channels (7) indeed model the pure dephasing noise.

Example 2. Consider now a slightly more restriced Hamiltonain, again for n spins 1/2,

$$H = \sum_{j<k} \left(J_{j,k}^x \sigma_j^x \sigma_k^x + J_{j,k}^y \sigma_j^y \sigma_k^y + J_{j,k}^z \sigma_j^z \sigma_k^z \right), \tag{9}$$

and up to $M = 2n$ Lindblad operators which represent incoherent local spin (particle) absorption (injection)

$$L_{2j-1} = a_j \sigma_j^+, \quad L_{2j} = b_j \sigma_j^-, \quad j = 1,\ldots,n. \tag{10}$$

Again, the interaction matrices $J_{j,k}^{x,y,z}$ and the spin-flipping rates a_j,b_j can be arbitrary. The Lemma can now be implemented with

$$U = \prod_{j=1}^{n} \sigma_j^y, \quad W = \prod_{j=1}^{n} \sigma_j^x. \tag{11}$$

In the XXZ-like case $J_{j,k}^x \equiv J_{j,k}^y$ the model represents an open interacting hard-core boson model. For nearest-neighbor interaction corresponding to one dimension (i.e., a chain), and if incoherent processes are only at the ends, $a_j = b_j = 0$, $2 \leqslant j \leqslant n-1$, the model also describes an open version of the so-called t-V model of spinless fermions.

In both examples above, the conditions (i)–(iii) of the Lemma are straightforward to check with the reflection matrix being trivial, $Z = \mathbb{1}_M$. Besides the necessary conditions $[H,U] = 0$, $[H,W] = 0$, implied by (i), we may or may *not* have $[U,V] = 0$; hence the number of spins n in our examples need not be even.

Provided the parameters in the Hamiltonian can be chosen such that both the energy spectrum and the energy-difference (frequency) spectrum are nondegenerate, one should observe aspontaneous $\mathbb{PT}$-symmetry-breaking transition of Liouvillian decay rates while increasing the noise or dissipation strength [10], and below the transition point all the coherences should decay with a uniform rate. It is reasonable to expect that both classes of examples should be relevant for experiments with ultracold atom systems.

We close this Brief Report by pointing out the fact that our explicit construction (the Lemma above) improves the statement on the symmetry of the dissipator-perturbation matrix V made in Eq. (14) of Ref. [10], which in general seems to be inconclusive. The symmetry of V is in fact needed in order to establish a nontrivial sponaneous $\mathbb{PT}$-symmetry-breaking transition (i.e., nonvanishing of $\gamma_{\rm PT}$). In order to prove that $V_{j,k} = V_{k,j}$, we only need to show that $\sum_m |\langle\psi_j|L_m|\psi_k\rangle|^2 = \sum_m |\langle\psi_k|L_m|\psi_j\rangle|^2$. Indeed, due to the condition (i) of the Lemma the eigenvectors $|\psi_j\rangle$ of H can be chosen to be simultaneously eigenvectors of W, say $W|\psi_j\rangle = \omega_j|\psi_j\rangle$, with $\omega_j \in \{\pm 1\}$, so we have

$$\begin{aligned}
&\sum_m |\langle\psi_j|L_m|\psi_k\rangle|^2 \\
&\quad = \sum_m |\langle\psi_j|W L_m|\psi_k\rangle|^2 \\
&\quad = \sum_{m,m',m''} Z_{m,m'} Z_{m,m''} \langle\psi_j|L_{m'}^\dagger W|\psi_k\rangle \overline{\langle\psi_j|L_{m''}^\dagger W|\psi_k\rangle} \\
&\quad = \sum_{m'} |\langle\psi_j|L_{m'}^\dagger|\psi_k\rangle|^2 = \sum_m |\langle\psi_k|L_m|\psi_j\rangle|^2,
\end{aligned}$$

since Z is orthogonal, $Z^T Z = \mathbb{1}_M$. ■

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