Evolution of entanglement under echo dynamics

Tomaž Prosen,[1] Thomas H. Seligman,[2] and Marko Žnidarič[1]
[1]*Physics Department, FMF, University of Ljubljana, Ljubljana, Slovenia*
[2]*Centro de Ciencias Físicas, University of Mexico (UNAM), Cuernavaca, Mexico*
(Received 8 April 2002; published 25 April 2003)

Echo dynamics and fidelity are often used to discuss stability in quantum-information processing and quantum chaos. Yet fidelity yields no information about entanglement, the characteristic property of quantum mechanics. We study the evolution of entanglement in echo dynamics. We find qualitatively different behavior between integrable and chaotic systems on one hand and between random and coherent initial states for integrable systems on the other. For the latter the evolution of entanglement is given by a classical time scale. Analytic results are illustrated numerically in a Jaynes-Cummings model.

DOI: 10.1103/PhysRevA.67.042112 PACS number(s): 03.65.Yz, 03.65.Sq, 05.45.Mt

More than a century ago Loschmidt, in his discussions with Bolzmann illustrating irreversibility in a gas, suggested to invert the velocity of each atom individually in order to revert to the initial situation. Recently, Loschmidt echoes have been of great interest in the control of quantum-information processing [1]. As entanglement is the essential property of quantum mechanics, in the present paper we analyze the Loschmidt echo appearing in the entanglement of two quantum systems. As a measure of entanglement we use purity [2] and show analytically as well as numerically that for classically integrable systems the purity decays as $1-c_{\mathrm{In}}t^2$, whereas for a classically chaotic system the decay after the Zeno time scale is described by $1-c_{\mathrm{ch}}t$. For a coherent state the constant c_{In} is independent of $\hbar$. For chaotic systems (or random initial states in integrable systems) this constant on the other hand is proportional to $1/\hbar^2$, thus defining quite different time scales.

Zurek [2] proposed to use the rate of decoherence as a characteristic of chaos in quantum mechanics and this is occasionally reinterpreted in terms of entanglement [3] between different parts of a closed quantum system. Such studies were limited to forward time evolution. Yet the insensitivity of quantum mechanics to small changes in initial conditions has been a basic difficulty in the introduction of the concepts of chaos to this field, and the idea to use sensitivity to perturbations in the time evolution [4] has emerged recently as one of the tools of choice to overcome this difficulty [5,6]. Specifically all these authors use *fidelity*, i.e., the correlation function between a quantum state evolving under the action of two Hamiltonians differing by a perturbation, which is equivalent to the autocorrelation function under echo dynamics. Since fidelity measures irreversibility of a full quantum state under the echo, it is also desirable to undertand irreversibility of a less restrictive quantity like entanglement. A recent study in spin chains [7], more related to quantum computing, revealed no qualitative difference between fidelity and the evolution of entanglement under echo dynamics as measured conveniently by purity there denoted as *purity fidelity*. This system allows no classical analog, and by consequence no coherent states. Yet these we shall show to be essential to recover results analogous to the ones of Zurek and Nemes [2,3]. Based on the idea that a partial trace simulates decoherence, our results lead to the conjecture that Zurek's result holds exclusively for coherent states in the central system, i.e., that we may not expect faster decoherence for chaotic central systems than for integrable ones, if we use random states that are more relevant to quantum information.

To implement this idea we have to consider systems with at least two degrees of freedom to allow for entanglement and a well defined classical limit $\hbar\to 0$. The (unperturbed) Hamiltonian will contain a parameter permitting a transition from order to chaos, and will typically couple the two degrees of freedom. A perturbation is then defined to obtain a second similar Hamiltonian. We give general results for entanglement under echo dynamics starting from an initial product (disentangled) state. To illustrate our results we use the Jaynes-Cummings (JC) model [8]. The usual corotating (integrable) version of this model has great practical importance in atomic physics and illustrates the $\hbar$-independent evolution of entanglement for coherent states. For the arguments involving the chaotic dynamics we include counterrotating terms [3,10] to construct a toy model that allows for chaos. Even this model may not be entirely unrealistic for atoms in a Paul trap in a driven field, as standard papers seek conditions where this term is small [11].

For general considerations and analytic calculations techniques of linear response developed originally for the evaluation of fidelity [6] are extended to calculate purity fidelity in terms of time correlation functions of the perturbation. In the case of coherent states we could carry the evaluation of linear response one step further using it in a semiclassical framework that relates the decay rates directly to the stability matrix of the orbit along which the packet evolves.

We consider the unitary time evolution given by the echo operator $M_\delta(t)=U_\delta^\dagger(t)U(t)$. Here $U(t)$ is generated by some unperturbed Hamiltonian H as $U(t)=e^{-iHt/\hbar}$ and similarly $U_\delta(t)=e^{-i(H+\delta V)t/\hbar}$, where V is the perturbation with strength δ. It is useful to rewrite the echo operator as time-ordered product [6] in the interaction picture,

$$M_\delta(t)=\hat{\mathcal{T}}\exp[\mathrm{i}\Sigma(t)\delta/\hbar], \tag{1}$$

where $\Sigma(t):=\int_0^t V(\tau)d\tau$ with $V(t):=U^\dagger(t)VU(t)$. This operator shall act on a composite system with the Hilbert space $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$, consisting of two factors with dimensions N_1

and N_2, which we may look upon as a "central system" and an "environment." We are interested only in information about the subsystem 1 which is contained in a *reduced density matrix* $\rho_1(t) := \mathrm{tr}_2 \rho(t)$, $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$. We shall study the purity fidelity [7] $F_P(t) := \mathrm{tr}_1[\rho_1(t)]^2$ as a measure of factorizability of a joint state $|\psi(t)\rangle = M_\delta(t)|\psi(0)\rangle$. We choose this quantity rather than some entropy because of its simple analytic dependence on $\rho_1(t)$. Here the partial traces with indices 1 and 2 are taken in the corresponding factor spaces and the reduced density matrix is acting on the first factor space. We always assume that we start with a factorized state at $t=0$, i.e., $F_P(0)=1$. For comparison we shall also use the fidelity $|F(t)|^2 = |\langle\psi(0)|M_\delta(t)|\psi(0)\rangle|^2$.

Expanding the echo operator (1) in δ, we get [6]

$$|F(t)|^2 = 1 - \delta^2 \hbar^{-2} C(t) + \cdots,$$

$$C(t) := \langle \Sigma^2(t)\rangle - \langle \Sigma(t)\rangle^2. \tag{2}$$

Here $\langle\cdot\rangle$ denotes an expectation in the product initial state $|\psi(0)\rangle = |1,1\rangle$ with the abbreviation $|i,\nu\rangle := |i\rangle_1 \otimes |\nu\rangle_2$. The same techniques yield for purity fidelity

$$F_P(t) = 1 - 2\delta^2\hbar^{-2}\{C(t) - D(t)\} + \cdots,$$

$$D(t) := \sum_{\nu\neq 1} |\langle 1,\nu|\Sigma(t)|1,1\rangle|^2 + \sum_{i\neq 1} |\langle i,1|\Sigma(t)|1,1\rangle|^2. \tag{3}$$

For both series to converge it is sufficient to use a *bounded* perturbation operator V, but we expect the linear-response formula to be a good approximation for a much wider class of perturbations. The somewhat unusual correlation function $D(t)$ contains only off-diagonal matrix elements of the operator $\Sigma(t)$ and determines the difference between $F_P(t)$ and $|F(t)|^4$. From expansions (2) and (3) we can see that the decay is determined by time correlation functions of the perturbation. The stronger the decay of correlation functions $\langle\psi|V(t)V(t')|\psi\rangle$ as $|t-t'|$ grows, the slower is the increase of $C(t)$ and the slower is the decay of $F(t)$ and $F_P(t)$.

We limit our discussion to systems which have a classical limit. For such systems chaos typically implies decay of the time correlation functions of the perturbation observable (i.e., mixing), while regular motion implies nonergodic behavior. Fidelity decay for both situations is discussed in Ref. [6]. Under rather general assumptions one finds *exponential* decay for *chaotic* dynamics

$$|F(t)|^2 = \exp(-t/\tau_{\mathrm{em}}), \quad \tau_{\mathrm{em}} = (2\sigma)^{-1}\hbar^2\delta^{-2}, \tag{4}$$

where a *diffusion coefficient* $\sigma := \lim_{t\to\infty} C(t)/(2t)$ is independent of the initial state $|\psi(0)\rangle$ (for sufficiently long times, typically $t \gg \ln 1/\hbar$). In classically *regular* situation the fidelity exhibits a quadratic decay in the leading order in δ even for long times, since $C(t) \to \bar{c}t^2$, where $\bar{c}$ depends on the structure of the initial state. For a coherent initial state we find a *Gaussian* decay of fidelity

$$|F(t)|^2 = \exp[-(t/\tau_{\mathrm{ne}})^2], \quad \tau_{\mathrm{ne}} = \bar{c}^{-1/2}\hbar\delta^{-1} \tag{5}$$

with $\bar{c} \propto \hbar$ [6]. It is worth to stress that in the regime of linear response (small δ), formulas (4) and (5) agree with Eqs. (2) and (3) from which the time scales $\tau_{\mathrm{em}}, \tau_{\mathrm{ne}}$ are obtained.

As for purity fidelity of chaotic systems, one may argue that $\Sigma(t)$ should look like a *random matrix* so the term $D(t)$ should be small compared to $C(t)$ in Eq. (3), namely, $D(t)/C(t) \sim 1/N_1 + 1/N_2$ because of the smaller number of terms involved in the sums. Thus, if both dimensions $N_{1,2}$ grow as $\hbar \to 0$ one expects in the asymptotic regime that $F_P(t) = |F(t)|^4 = \exp(-2t/\tau_{\mathrm{em}})$, and does not significantly depend on the initial state. In particular this also holds for coherent initial states. Using similar arguments for regular dynamics but a random initial state, one again sees that $F_P(t)$ follows $|F(t)|^4$ closely [7].

Yet for *coherent states* and regular classical dynamics this is *not* the case because the term $D(t)$ is not negligible. We show that the difference $C(t) - D(t)$ cancels in the leading order in $\hbar$, i.e., $C(t) - D(t) \sim \hbar^2$, meaning that $F_P(t)$ as compared to $|F(t)|^2$ decays on a qualitatively longer, $\hbar$-independent time scale $\tau_{\mathrm{ne}}^P = K/\delta \sim \hbar^{-1/2}\tau_{\mathrm{ne}}$. This will be the main result of the present paper. In order to establish this we consider the evolution of a Gaussian wave packet along a stable orbit $\vec{z}_t = (\vec{x}_t, \vec{p}_t)$ as $\langle\vec{x}|\psi(t)\rangle = C\exp((i/\hbar)[(\vec{x}-\vec{x}_t)\cdot A_t(\vec{x}-\vec{x}_t) + \vec{p}_t\cdot\vec{x}])$, where the block form of the complex $d\times d$ matrix

$$A_t = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{6}$$

corresponds to obvious division of $(d = d_1 + d_2)$-dimensional configuration space into d_1- and d_2-dimensional parts. A_t is a ratio of two pieces of a classical monodromy matrix [9] so it is $\hbar$ independent. The purity of a reduced wave packet $\rho(x_1, x_1') = \int dx_2 \langle x_1, x_2|\psi(t)\rangle\langle\psi(t)|x_1', x_2\rangle$ is $F_P = \int dx_1 dx_1' |\rho(x_1, x_1')|^2 = 1$ if $A_{12} = A_{21} = 0$ while in general we find $\hbar$-independent expression

$$F_P = (\det \mathrm{Im} A) \times \begin{vmatrix} \mathrm{Im} A_{11} & \frac{\mathrm{i}}{2}A_{12}^* & 0 & -\frac{\mathrm{i}}{2}A_{12} \\ \frac{\mathrm{i}}{2}A_{21}^* & \mathrm{Im} A_{22} & -\frac{\mathrm{i}}{2}A_{21} & 0 \\ 0 & -\frac{\mathrm{i}}{2}A_{12} & \mathrm{Im} A_{11} & \frac{\mathrm{i}}{2}A_{12} \\ -\frac{\mathrm{i}}{2}A_{21} & 0 & \frac{\mathrm{i}}{2}A_{21}^* & \mathrm{Im} A_{22} \end{vmatrix}^{-1/2},$$

where $|\cdot|$ denotes a determinant of $2d \times 2d$ matrix. For classical echo dynamics, the covariance matrix $A = A_t$ is given by a linear stability analysis as $A = A_0 + t\delta B$ for some matrix B, where $A_{0,12} = A_{0,21} = 0$. Then purity fidelity is $\hbar$ independent and can be evaluated in the leading orders as $F_P(t) = 1 - (t\delta/K)^2 + \cdots$.

We thus reach the following interesting conclusion: Both fidelity and purity fidelity decay quadratically in integrable situations, while they decay linearly in chaotic ones, once we are beyond the Zeno time scale. Yet there is a very relevant difference in time scales themselves, if we discuss the purity of coherent rather than random states. For integrable systems, purity fidelity decays on an $\hbar$-independent scale. This leads to situations with very stable purity fidelity, while the same perturbation generates decay of the fidelity of the coherent state as well as the decay of the purity fidelity of a random state on much shorter time scales, dictated by the value of $\hbar$. Note though that for sufficiently small perturbations at fixed $\hbar$ the quadratic decline of purity fidelity always prevails.

To illustrate these results we use the JC Hamiltonian including corotating *and* counterrotating terms for the chaotic case as

$$H=\hbar\omega a^{\dagger}a+\hbar\epsilon J_{\mathrm{z}}+\frac{\hbar}{\sqrt{2J}}(GaJ_{+}+G'aJ_{-}+\mathrm{H.c.}) \quad (7)$$

with standard boson operators $a,a^{\dagger}$, $[a,a^{\dagger}]=1$, and standard SU(2) generators $J_{\pm},J_{\mathrm{z}}$. We choose $\hbar=1/J$ ensuring that the classical limit is reached for $J\to\infty$ while the angular momentum $\hbar J=1$ is fixed. If either $G=0$ or $G'=0$ the model is integrable with an additional invariant being the difference or the sum of quanta for the spin and the oscillator. In all calculations we used coherent initial states for the product system, i.e., direct product of coherent states of the oscillator, $|\alpha\rangle_2=e^{\alpha a^{\dagger}-\alpha^* a}|0\rangle_2$, and of the spin [SU(2)], $|\theta,\phi\rangle_1=(1+\tau\tau^*)^{-J}\exp(\tau J_-)|J,J\rangle$ with $\tau=e^{\mathrm{i}\phi}\tan(\theta/2)$ [12].

For our numerics we fix $J=4$ and choose initial position of SU(2) coherent state at $(\theta,\phi)=(1,1)$ and for the oscillator at $\alpha=1.15$. The parameters in JC Hamiltonian are (a) in chaotic regime $\omega=\epsilon=0.3$ and $G=G'=1$, (b) in integrable regime $\omega=\epsilon=0.3$ and $G=1,G'=0$. The corresponding classical Poincaré section shows a single practically ergodic component in the chaotic case (a) (at energy $E=1.0$ determined from the initial condition), whereas integrable case (b) ($E=0.63$) shows a generic family of invariant tori. The perturbation is realized by varying the parameter ϵ in JC Hamiltonian (7), also known as dephasing, so the (bounded) perturbation generator is $V=\hbar J_{\mathrm{z}}$.

We now show numerical results obtained by diagonalization in truncated Hilbert spaces. Stability of the calculation with respect to truncation was checked. Figure 1 presents the correlation integrals $C(t)/t$ and $D(t)/t$ for chaotic and regular regimes. For chaotic dynamics [case (a)] the correlation integral converges after $t\approx 10$ to a well defined diffusion coefficient $\sigma=0.10$ with the D term being of order of $1/N_1+1/N_2\approx 1/4$. For regular dynamics [case (b)] and $t>10$ the correlation integral grows as t^2 due to a nonvanishing plateau $\bar{c}=0.046$ in the correlation function. In this case the difference $C(t)-D(t)$ is approximately $C(t)/J\propto\hbar^2$, which has been checked numerically also for larger $J\leqslant 24$, confirming $\hbar$-independent decay of $F_P(t)$. The oscillations in these functions are not accounted for by the present theory, and are probably particular but interesting properties of the model. Whether they relate to oscillations seen in Ref. [3] is an open question.

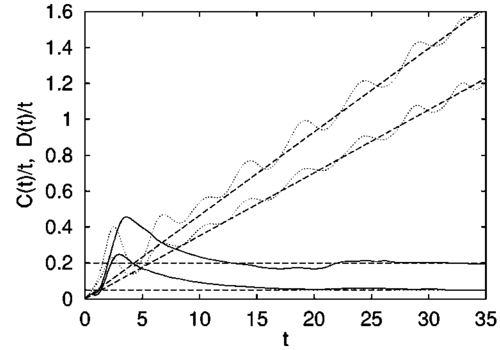

FIG. 1. Correlation integrals $C(t)/t$ and $D(t)/t$ for the regular (top two dotted curves) and chaotic (lower two solid curves) regimes. In both cases the upper curve is for $C(t)/t$ and the lower for $D(t)/t$. The horizontal dashed lines indicate $2\sigma=0.20$ (upper) and 0.20/4 (lower), whereas the increasing ones have the slopes $\bar{c}=0.046$ (upper) and $(1-0.98/4)\bar{c}$ (lower).

We first report a calculation with a strong perturbation $\delta=0.1$, which rapidly exceeds the realm of validity of linear response, in Fig. 2 where the main figure gives the purity fidelity and the inset the fidelity. For the fidelity decay (inset) we find excellent agreement with the exponential decay (4) in a chaotic regime and a faster Gaussian decay in a regular regime (5), where the decay rates are fixed as above. However, for purity fidelity we find already at $t\approx 20$ that the decay starts to be influenced by the saturation value of $F_P(t\to\infty)\approx 1/(2J+1)$. Therefore purity fidelity is higher for the integrable case than for the chaotic one not only at short times, as expected, but even at large times. This is relevant because we shall next choose a weak perturbation $\delta=0.005$ to avoid this problem. We expect and find the crossover after a fairly short time. This calculation allows comparison with theory as well as an illustration of the evolution of the square of the Wigner function, corresponding to the reduced density matrix $\rho_1(t)$ for the angular-momentum states on the sphere using the definition of Ref. [12]. Near the top and bottom of Fig. 3 we see this evolution for the chaotic and the integrable Hamiltonian, respectively. In the center of the figure, we plot the purity fidelity on the same

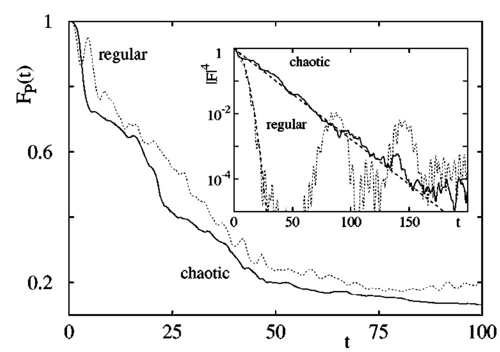

FIG. 2. Purity fidelity $F_P(t)$ (main figure) and squared fidelity $|F(t)|^4$ (inset) in the chaotic regime (solid curves) and in the integrable regime (dotted curves), for $\delta=0.1$. The dashed lines indicate the linear and quadratic approximations, respectively. Note the differences in vertical scales.

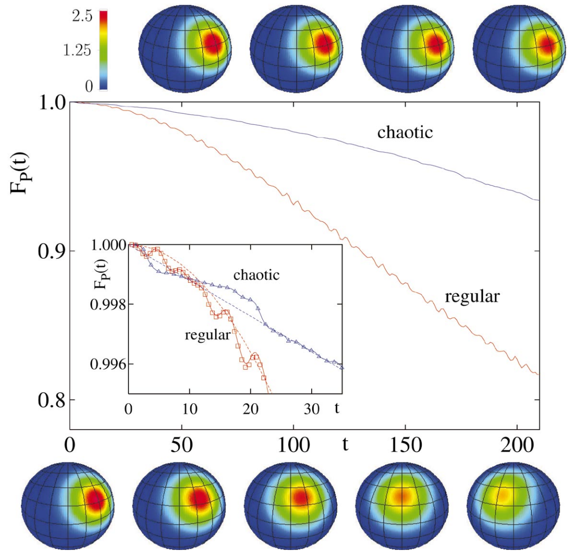

FIG. 3. (Color) Echo dynamics for weak coupling: $\delta = 0.005$. Square of the Wigner function for chaotic dynamics (top diagrams) and integrable dynamics as a function of time (bottom diagrams) at times corresponding to the axis. Color code: top left. Purity fidelity is shown in the frame on the same time scale and for short times in the inset. Red curves give the integrable and blue curves the chaotic evolution. Full curves show the complete numerics, symbols the evaluation starting from the numerical correlation functions of Fig. 1, and dashed curves the linear or quadratic approximation.

time scale as the Wigner functions in the main frame and an amplification of short times in the inset. We observe detailed agreement of numerics with results obtained from the numerical values of the correlation integrals (2) and (3) reproducing the oscillatory structure. From the same correlation integrals we obtained the coefficients for the linear and quadratic decays, which agree well if we discard the oscillations. We see a crossing of the two curves at $t = t_P^* \approx 12$ for F_P. These times are larger than the Zeno time (≈ 1) and indicate the competition of the decay rate and the decay shape as expected for a nonsmall value of $\hbar = 1/4$. It is important to remember that the integral over the square of the Wigner function gives the purity and therefore the fading of the picture will be indicative of the purity decay. On the other hand, the movement of the center is an indication of the rapid decay of fidelity (not shown in the figure).

In this paper, we study the linearized behavior of the evolution of entanglement under echo dynamics for time scales large compared to those of the quantum Zeno effect, but sufficiently short for the expansion to be valid. Similarly to the behavior of fidelity the decay of purity fidelity is typically quadratic for nonmixing systems, and linear for mixing ones, the first situation arising for integrable systems and the second for chaotic ones. An interesting particular but relevant case appears if we consider coherent initial states and integrable classical dynamics. In this case we have shown that purity fidelity, still having a quadratic decay, can be computed classically in the leading order which is $\hbar$ independent, so the time scale for purity fidelity decay of a coherent state is longer by a factor proportional to $\hbar^{-1/2}$ than the corresponding one for a random state. Coherent states in integrable systems are thus particularly long lived for semiclassical echo situations. On the other hand, for chaotic classical dynamics and coherent initial states we find that purity fidelity is the same as for random states, and its decay will be slower than for either random or coherent initial states and integrable dynamics provided that time is sufficiently long or perturbation δ is sufficiently small.

We are grateful to W. Schleich for his extensive discussion of our manuscript. Financial support by the Ministry of Education, Science and Sports of Slovenia and from projects under Grant No. IN-112200, DGAPA UNAM, Mexico, Grant No. 25192-E CONACYT Mexico, and Grant No. DAAD19-02-1-0086, ARO United States is gratefully acknowledged. T.P. and M.Ž. thank CIC, where this work was completed, for its hospitality.

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