Entanglement and random quantum states

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Abstract. Motivated by the fact that random quantum states are generic, i. e., that they occur as eigenvectors or during time evolution of sufficiently complex quantum system, we are going to address three questions: (i) what is the entanglement content of random quantum states, (ii) how can we efficiently generate them, and (iii) how come that there is apparently no entanglement in macroscopic world. To explain the later we will use properties of random quantum states. To answer the first question we are going to calculate average values of all Schmidt coefficient and to address the second one we are going to explicitly calculate convergence times of various random protocols.

Keywords: random quantum states, entanglement, decoherence, Markov chain
PACS: 03.65.Ud, 03.67.-a, 03.65.Yz

INTRODUCTION

Quantum entanglement is one of the properties that makes quantum mechanics "strange" and very different from classical physics. While the concept of entanglement has been known from the very beginning of quantum mechanics, only relatively recently have experiments advanced to the point of being able to manipulate individual quanta. What is more, it has been realized that the entanglement can be used to our advantage and is a crucial resource for many quantum information processes. It enables for powerful quantum protocols that are better than the best classical procedures. For instance, certain problems can be solved in less steps by using quantum algorithms than with classical ones and for this speedup sufficient entanglement is needed during quantum computation. Understanding the nature of entanglement is therefore of paramount importance, for a review see, e. g. [1]. Characterizing entanglement of arbitrary mixed states is a difficult and still unsolved problem. Because of that one frequently focuses on entanglement properties of some simpler but nevertheless important states. Here we are going to focus of random quantum states, that is on an ensemble of states distributed according to unitarily invariant measure. The reason to choose random quantum states in not just their simplicity but also their importance. They are typical in a sense that statistical properties of almost all states from Hilbert space are well described by those of random quantum states. From quantum chaology we also know that the same holds true for eigenvectors of sufficiently complex quantum systems and that time evolution in a chaotic system will typically produce random states [2]. Because random quantum states possess a large amount of entanglement they are useful in certain quantum information processes like quantum dense coding and remote state preparation [3, 4].

While entanglement is a desired property in microscopic systems employed in quantum information, there is apparently a complete lack of it, or at least no observable

CP1076, *Let's Face Chaos Through Nonlinear Dynamics: 7th International Summer School and Conference*
edited M. Robnik and V. Romanovski

manifestation, in macroscopic objects at high temperature. There have been many attempts to explain this classicality of macroscopic systems. Most notably, decoherence due to external degrees of freedom is usually credited as being responsible for the disappearance of entanglement from macroscopic superpositions. In a nutshell, the argument goes as follows: even if the system is in an entangled state at the beginning, e. g., in a coherent superposition of two macroscopic states, time evolution will in general transform this coherent superposition into an incoherent on, i. e., into a classical mixture. The reason for such decoherence is an always present residual coupling of our central system to many uncontrollable external degrees of freedom – the environment. For a review of decoherence see [5]. However, one must be aware that the evolution of the central system plus environment is still unitary and therefore, even though the final state of the central system and environment will presumably be very complex, it will be a pure state possessing some bipartite entanglement. How come that we do not observe any manifestation of this entanglement? As we will see, some of these issues can be resolved by using properties of typical that is random quantum states.

With all these in mind we are going to study the entanglement properties of random quantum states. Specifically, in the first section we are going to quantify the entanglement content of such states by calculating the average values of all Schmidt coefficients. In quantum information one wants quantum protocols which are faster than the best classical ones. One is therefore interested in how the time needed scales with the size of the problem. In the second section we are going to study how fast can one generate random quantum states. This is a very relevant question as one could argue that studying random quantum states makes sense only if they can be prepared efficiently. We will see that they indeed can be prepared efficiently as long as bipartite entanglement is concerned. Using entanglement properties of random quantum states the third section will present arguments why there are no observable manifestations of entanglement in macroscopic systems. There will be two issues, one is the difficulty of detecting entanglement, even if it is present, and the second one will be the role played by generic initial conditions. As we will see the initial randomness very efficiently wipes out the entanglement thereby acting as an internal source of decoherence. The work presented below has already been published in journal papers.

ENTANGLEMENT CONTENT OF RANDOM STATES

Entanglement of pure quantum states is completely specified by Schmidt coefficients μ_i, defined through the decomposition

$$|\psi\rangle = \sum_{i=0}^{N_{\mathrm{A}}-1} \mu_i |w_i^{\mathrm{A}}\rangle \otimes |w_i^{\mathrm{B}}\rangle, \tag{1}$$

where $|w_i^{\mathrm{A}}\rangle$ and $|w_i^{\mathrm{B}}\rangle$ are mutually orthonormal on the respective subspaces $\mathscr{H}_{\mathrm{A}}$ and $\mathscr{H}_{\mathrm{B}}$ of dimensions $N_{\mathrm{A}} = 2^{n_{\mathrm{A}}}$ and $N_{\mathrm{B}} = 2^{n_{\mathrm{B}}}$, total Hilbert space dimension being $N = N^{\mathrm{A}}N^{\mathrm{B}} = 2^n$. We assume that Schmidt coefficients are ordered in a non increasing way, $\mu_0 \geq \mu_1 \geq \cdots \geq \mu_{N_{\mathrm{A}}-1}$. Squares of Schmidt coefficients, $\lambda_i = \mu_i^2$, are nothing but the eigenvalues of the reduced density matrix, $\rho_{\mathrm{A}} = \mathrm{tr}_{\mathrm{B}}|\psi\rangle\langle\psi|$. In all our discussions $|\psi\rangle$

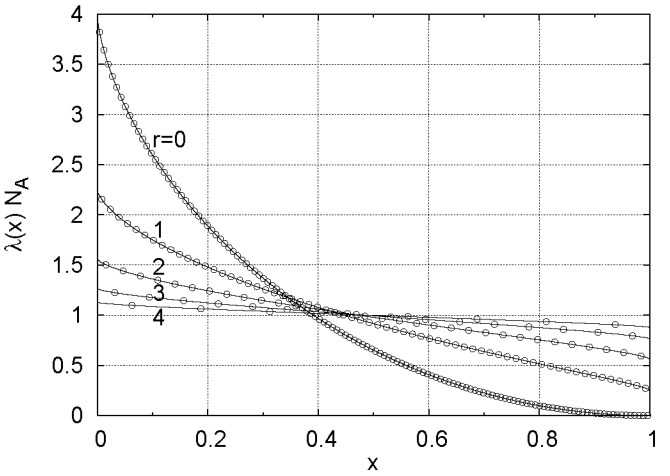

FIGURE 1. Scaled eigenvalues $\lambda(x)$ for a bipartite cut to $n/2-r$ and $n/2+r$ qubits, $n_{\rm B}-n_{\rm A}=2r$. Circles are numerics for $n=14$ while the full line is theoretical $f(x)$ (3). From [6].

are going to be random quantum states. They can be obtained in various equivalent ways, for instance, as eigenvectors of random unitary matrices obtained by, e. g., using Hurwitz parameterization or simply by randomly drawing expansion coefficients according to Gaussian distribution [7]. Average values of various scalar functions of λ_i for random quantum states, like for instance von Neumann entropy or purity, have been calculated before [7, 8, 9]. Here we are going to calculate average values of λ_i, $\langle\lambda_i\rangle$. Because we are interested in the large N limit and because λ_i scale as $\sim 1/N_{\rm A}$, we are going to define a continuous function $f(x)$ which will give us the average λ_i in the limit $N_{\rm A}\to\infty$,

$$\langle\lambda_i\rangle=\frac{1}{N_{\rm A}}f(x),\qquad x=\frac{i+1/2}{N_{\rm A}},\quad i=0,\ldots,N_{\rm A}-1. \tag{2}$$

While one can write an explicit expression for a joint probability distribution of all eigenvalues λ_i [7, 9], using it to calculate for instance the average values of λ_i is not easy. We rather take a different route. In the limit of large dimensions one knows that the distribution of eigenvalues is given by the Marčenko-Pastur law [10]. Because $f(x)$ is a monotone function of x we can write down a simple differential equation, whose solution gives $f(x)$ in an implicit form [6],

$$\begin{aligned} f(x) &= a+(b-a)\cos^2\varphi, \qquad \{N_{\rm A}\le N_{\rm B}\} \\ \frac{\pi}{2}x &= \frac{1+w}{2w}\varphi-\frac{1}{2\sqrt{w}}\sin(2\varphi)-\frac{1-w}{2w}\arctan\left(\sqrt{\frac{a}{b}}\tan\varphi\right), \end{aligned} \tag{3}$$

where a,b and w are determined by Hilbert space dimensions as

$$a=(1-\sqrt{w})^2,\qquad b=(1+\sqrt{w})^2,\qquad w=\frac{N_{\rm A}}{N_{\rm B}}=\frac{1}{2^{n_{\rm B}-n_{\rm A}}}. \tag{4}$$

The form of function $f(x)$ for various bipartite splittings can be seen in Fig. 1. With the increasing difference between dimensions of subspaces A and B the spectrum gets increasingly flat. For equal bipartition, $N_A = N_B$, the largest eigenvalue is $4/N_A$ while the smallest is exactly $1/N_A^3$ (this was conjectured in [6] and proved in [11]). The distribution of individual eigenvalues λ_i becomes Gaussian for large dimensions with the width scaling as $\delta\lambda_i \sim 1/N_A^2$, i. e., relative width $\delta\lambda_i/\lambda_i$ goes to zero as $\sim 1/N_A$ [6]. Correlations between λ_i are also small so that in the leading order in $1/N_A$ they can be considered as independent gaussian random numbers with the mean given by eq. (3). Recently the distribution of λ_i has been derived for finite sizes [12].

EFFICIENT QUANTUM PROTOCOLS TO GENERATE RANDOM STATES

An important question from a practical point of view is how can we generate random quantum states? One possible procedure is the following: generate some pseudo-random sequence of gates, thereby producing the so-called pseudo unitary operator, applying it to an arbitrary separable initial state. After a sufficient number of steps we will end up with a random state. Each step of a protocol consists of a single application of a two qubit gate applied to a random pair of qubits. Such random protocol has been numerically studied in [13] while a convergence to a uniform Haar measure has been discussed in [14]. Of particular importance is the question whether the random protocol is efficient, that is, if the number of steps scales polynomially with the number of qubits n or not. Of course, one should be aware that to produce an arbitrary unitary transformation, and therefore a truly random state, an exponential number of two qubit gates is needed. However, if our criteria is just to reproduce a bipartite entanglement of a typical random state, which is the case in our discussions, only polynomial number of gates is needed.

Recently, the question of how many gates do we need to obtain convergence has been studied by analytical tools of Markovian chains [15]. Transformation to Markovian chains goes as follows (for details see original derivation [15]). First, we expand density matrix $\rho = |\psi(t)\rangle\langle\psi(t)|$ over products of local Pauli matrices, $\rho = \sum_\alpha c_\alpha \, \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$. As a measure of bipartite entanglement we use purity, which is given by $I(t) = \mathrm{tr}_A[\rho_A^2(t)] = \frac{1}{N^2}\sum_{\alpha=\{\alpha_A 0_B\}} c_\alpha^2(t)$. Here $\sigma_i^{\alpha_i}$ denotes Pauli matrix $\alpha_i \in \{0,x,y,z\}$ acting on ith qubit, with the convention $\sigma^0 = \mathbb{1}$. We want to calculate time dependence of purity for a protocol consisting of application of a random two-qubit gate $U_{ij}(t)$ acting on ith and jth qubits at step t, $|\psi(t+1)\rangle = U_{ij}(t)|\psi(t)\rangle$. $U_{ij}(t)$ is going to be a product of random single qubit gates $V(t)$ and $V'(t)$, independent for each qubit and at each step, and a fixed two qubit gate W, $U_{ij}(t) = V_i(t)V'(t)_j W_{ij}$. Averaging over random single qubit unitaries V and V' from $U(2)$ one can arrive at the transformation law of coefficients c_α after one step of the protocol. If a two qubit gate W preserves products of Pauli matrices, i. e., if W transforms a product of two Pauli matrices into a product of some other two Pauli matrices (apart from a phase), then the transformation can actually be written for c_α^2,

$$c^2(t+1) = M\,c^2(t), \qquad M = \frac{1}{L}\sum_{ij} M_{ij}^{(2)}, \tag{5}$$

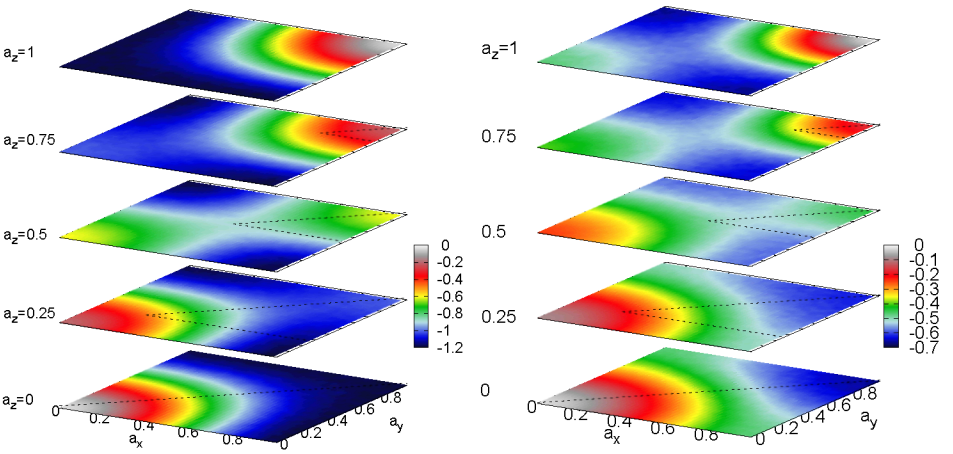

FIGURE 2. Decay rate of purity for different parameters $a_{x,y,z}$ (6) and random protocol (black areas denote the fastest decay). On the left is for gates between all pairs of qubits and on the right for nearest-neighbor gates. From [16].

where by c^2 we denote a vector with components c_α^2. Two qubit matrix $M^{(2)}$ depends only on the two qubit gate W used in the protocol. The sum in definition of Markovian matrix M goes over all allowed couplings $i-j$, i. e., depending on the protocol this can be nearest neighbor qubits with open or periodic boundary conditions or all pairs of qubits. Markov matrix M has two eigenvalues equal to 1, corresponding to invariant states being an identity operator and a uniform mixture of all basis states, and a third eigenvalue $1-\Delta$, where Δ is the gap of Markovian matrix M. If this eigenvalue is nondegenerate purity will asymptotically decay as $I(t) \approx \exp(-t\Delta)$ and will reach level $I(\tau) = \varepsilon$ after convergence time $\tau \sim \frac{1}{\Delta} \ln \frac{1}{\varepsilon}$. To access convergence time of random protocol one therefore has to calculate or estimate the gap Δ. This has been done in [15] where it has been proved that $\Delta \geq\sim 1/n^2$, meaning that purity will reach its asymptotic level $I(\infty) \sim 1/2^{n/2}$ after time $\sim n^3$.

In a subsequent numerical investigation [16] a two qubit gate has been identified for which the decay of purity is the fastest. Up-to local single qubit rotations every two qubit gate can be represented in its canonical form [17]

$$w(a_x, a_y, a_z) = \exp\left(\mathrm{i}\frac{\pi}{4}[a_x \sigma_i^x \sigma_j^x + a_y \sigma_i^y \sigma_j^y + a_z \sigma_i^z \sigma_j^z]\right), \tag{6}$$

with three parameters $a_{x,y,z}$. In Fig. 2 one can see the dependence of the decay rate of purity on three parameters. For protocol with gates between all pairs of qubits the fastest decay is achieved for CNOT as well as for XY gate, while in the case when two qubit gates are allowed only between nearest neighbors the fastest convergence is reached for XY gate, see [16] for details. In Fig. 3 one can see that other quantities like Schmidt coefficients converge to the asymptotic values of random states (3) on equal time scale as

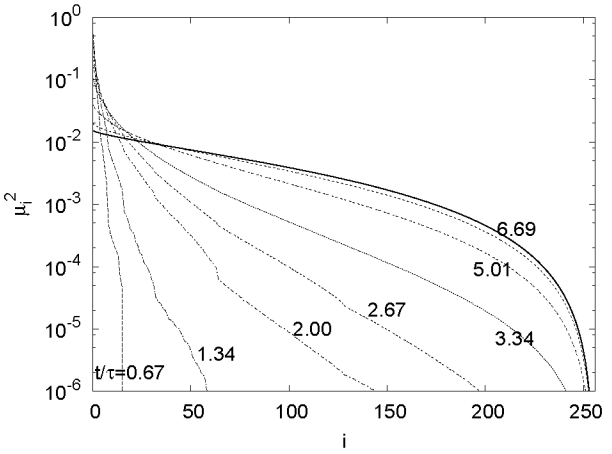

FIGURE 3. Convergence of Schmidt coefficients during random protocol. Time is expressed in units of the decay time of purity. Full line is theoretical prediction for random states (3). From [16].

purity decays. Regarding the scaling of the purity decay time on n there were numerical indications [16] that it scales as $\sim 1/n$ and not as $\sim 1/n^2$ as a bound proved in [15] would tend to suggest. This $1/n$ scaling was furthermore strengthened by an analytical derivation for random $U(4)$ gate and coupling between all pairs of qubits [18]. Note that one can use known facts about entanglement production in chaotic systems to argue [16] that the optimal scaling of the decay time should indeed be $\sim 1/n$.

It is possible to derive [19] an explicit expression for the gap of Markovian chain for various gates and couplings. Analytical results for the gap Δ are obtained by mapping an eigenvalue problem of Markovian chain exactly to that of various (depending on the two qubit gate) integrable spin chains. The result for CNOT as well as for XY gate between all pairs of qubits is

$$\Delta = \frac{4}{3n} + \mathcal{O}(1/n^2). \tag{7}$$

For completely random $U(4)$ gate between all pairs one gets slightly smaller gap,

$$\Delta = \frac{6}{5n} + \mathcal{O}(1/n^2). \tag{8}$$

For $U(4)$ and nearest-neighbor coupling with periodic or open boundary conditions the gap is

$$\Delta_{\mathrm{PBC}} = \frac{2(1-\frac{4}{5}\cos(\pi/n))}{n}, \quad \Delta_{\mathrm{OBC}} = \frac{1-\frac{4}{5}\cos(\pi/n)}{n-1}. \tag{9}$$

All these analytical expressions improve and extend results in [15, 18].

APPARENT LACK OF ENTANGLEMENT IN MACROSCOPIC SYSTEMS

In the next two subsections we are going to present two arguments, using random quantum states, as to why there is apparently no macroscopic manifestation of entanglement. Because the evolution of a system of interest and its environment is unitary the combined state is a pure one. Therefore, in general, there will be some bipartite entanglement present say between systems and environmental degrees of freedom. The question is then why can we not observe this entanglement.

The resolution of this apparent paradox is similar to the one with the second law of thermodynamics [20]. Increasing of the thermodynamic entropy with time is also seemingly in contradiction with the reversibility of the underlying equations of motion. For explanation one can use two observations: (i) practicality – performing time reversal by, e. g., reversing velocities of all particles might be close to impossible from a practical point of view; (ii) probability – initial conditions are prevalently of such form that for almost all the entropy will increase with time. Similar arguments will be used in the next two subsections to explain the apparent lack of entanglement in quantum systems with many degrees of freedom.

Practicality of detecting entanglement

Even though a joint pure state representing central system and environment is bipartite entangled, the detection of entanglement will be in general close to impossible because it would require a very complex measurement involving very many degrees of freedom [21]. For all practical purposes the detection of entanglement in such states is impossible. Let us look in more detail.

In Ref. [21] we studied the detection of entanglement in mixtures of independent random states $|\psi_i\rangle$,

$$\rho = \sum_{i=1}^{m} \frac{1}{m} |\psi_i\rangle\langle\psi_i|, \tag{10}$$

using an entanglement witness W [22]. By definition, an entanglement witness is such Hermitian operator W that $\mathrm{tr}\,(W\rho_{\mathrm{sep}}) \geq 0$ for all separable states ρ_{sep} while there exists at least one entangled state ρ_{ent} such that $\mathrm{tr}\,(W\rho_{\mathrm{ent}}) < 0$. Therefore, negative expectation value of W is a signature of entanglement. Because it is easier to measure larger absolute expectation value one can argue that the detection of entanglement is easier the larger this absolute value of negative expectation is. In the following we will limit ourselves to decomposable entanglement witnesses written as $W = Q^{\mathrm{T_B}}$, i. e., as a partial transposition of a positive operator Q, and we will always consider symmetric bipartition with subspaces A and B being of equal size.

Let us first consider the case when we want to detect entanglement in a single ($m = 1$ in eq. 10) unknown random state. The best strategy is to choose for Q a projector on another (independent) random state. One can show [21] that the probability distribution of measurement results, $w/N = \mathrm{tr}\,(W\rho)$, where N is the dimension of total Hilbert space,

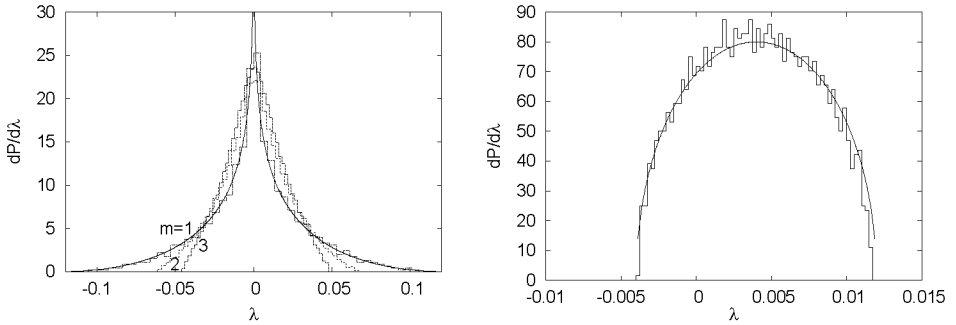

FIGURE 4. Distribution of eigenvalues of ρ^{T_B} (10) for mixtures of different number of random states m. Full curve in left plot is theory (12). In right plot $m = 256$. All is for $N = 2^{10}$. From [21].

converges to a Gaussian for large N,

$$p(w) = \frac{1}{\sqrt{2\pi}} \exp(-(w-1)^2/2). \tag{11}$$

Note that measurement results scale as $\sim 1/N$ and are therefore exponentially small in the number of qubits. To detect entanglement in a random state one must be able to perform measurements with an exponential precision. But even with the exponential precision the probability to detect entanglement in a random state is just $\int_{-\infty}^{0} p(w)dw = (1 - \mathrm{erf}(1/\sqrt{2}))/2 \approx 0.159$. If we instead try to detect entanglement in a mixture of m random states (10) probability is even smaller, for large m exponentially small in m, $\int_{-\infty}^{0} p(w)dw \asymp \frac{1}{\sqrt{2\pi m}} \mathrm{e}^{-m/2}$.

In case we know the measured random state in advance we can prepare an optimal W corresponding to the minimal eigenvalue of ρ^{T_B} (10). It is therefore interesting to look at the distribution of eigenvalues of ρ^{T_B}, where ρ is a mixture of m independent random states. For a single random state, $m = 1$, one can derive [21] an explicit expression for the distribution of scaled eigenvalues $y = \sqrt{N}\lambda$,

$$\frac{\mathrm{d}\mathscr{P}}{\mathrm{d}y} = \frac{1}{8\pi^2} \left[(16+y^2)K(1-y^2/16) - 32E(1-y^2/16) \right], \quad y \in [-4,4], \tag{12}$$

where $E(x)$ and $K(x)$ are elliptic integrals. For mixtures of more random states, $m > 1$, we present only results of numerical simulation in Fig. 4.

Role of generic initial conditions

In this subsection we are going to study how the evolution with a general Hamiltonian changes entanglement of a smaller subsystem, starting from a generic initial condition. The idea is that we are able to control and therefore also measure only a small number of total degrees of freedom (of system + environment) and therefore the relevant question

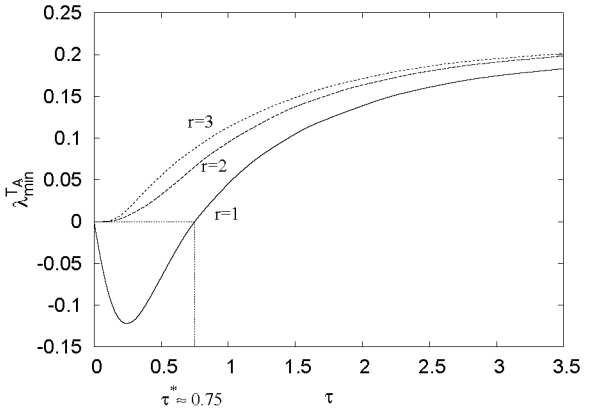

FIGURE 5. Average minimal eigenvalue of $\lambda_{\min}^{T_A}$ for a two-body random matrix model. Curves with different r show entanglement between nearest-neighbor qubits ($r = 1$), qubits separated by one qubit ($r = 2$) and those separated by two qubits ($r = 3$). Entanglement is present only between directly coupled qubits and only for short times $\tau < \tau^*$. From [23].

is how long and how much entanglement do we have within this smaller subsystem. In particular, we are going to study entanglement between two qubits during Hamiltonian time evolution of a larger homogeneous n qubit system of which these two central qubits are part of.

For each specific Hamiltonian, being either chaotic of integrable, there are initial conditions for which entanglement at some later time is arbitrary large. However, such initial conditions are very special and particular to each Hamiltonian. Our argument here is that provided one starts from a generic initial state possessing some randomness, in our case these will be product random states, the evolution of entanglement will be to a large extend independent of the Hamiltonian (as long as one sticks to physical ones, i. e., only few-body interactions) and therefore universal. This universality will come about due to randomness in the initial condition.

Our initial state will be a random product state on all qubits, with similar results obtained also for other choices [23]. Evolving such a state for sufficiently long time will in general results in a random state for which we know that tracing out $n-2$ qubits will, for large n, with high probability result in a separable two qubit reduced density matrix [24]. For small time though there will be two other competing effects. One is entanglement production due to time evolution with non-separable Hamiltonian, and the other is entanglement loss due to the spreading of initial-state randomness throughout the system and the approach of system's state to a random state. The net result will be the increase of entanglement at short times and a complete lack thereof after some finite critical time. In [23] we have numerically simulated various one dimensional spin Hamiltonians, obtaining a universal behavior in all cases (taking into account a non-universal characteristic time-scale for each Hamiltonian). This universal behavior can in turn be well modeled by a nearest-neighbor two-body random matrix model with Hamiltonian $H = \sum_i h_{i,i+1}$, where $h_{i,i+1}$ act non-trivially only on two qubits, for which

it is a 4×4 random hermitian matrix, same for all coupled pairs and normalized as $\mathrm{tr}(h_{i,i+1}^2) = 1$. A random hermitian matrix is a matrix whose matrix elements are independent random complex Gaussian numbers [25]. In Fig. 5 we show results of numerical simulation. We characterized entanglement by calculating the minimal eigenvalue of a partially transposed reduced density matrix of two qubits of interest. These two qubits have been chosen from the center of the chain and were separated by $r-1$ other (environmental) qubits. If $r = 1$ we are studying entanglement between qubits which are directly coupled by our Hamiltonian. We can see that two qubits are entangled only for short times and only if they are directly coupled by the Hamiltonian. For initial states possessing some randomness this is a general conclusion and holds regardless of the Hamiltonian used for time evolution [23]. One consequence of this is that it is hard to generate entanglement regardless of the dynamics if there is some randomness present in the initial state.

REFERENCES

1. R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, e-print quant-ph/0702225.
2. F. Haake, *Quantum Signatures of Chaos*, 2nd. ed. Springer Verlag, Berlin, 2001.
3. A. Harrow, P. Hayden, and D. Leung, *Phys. Rev. Lett.* **92**, 187901 (2004); A. Abeyesinghe et al., *IEEE Trans. Inform. Theory* **52**, 3635 (2006).
4. P. Hayden et al., *Comm. Math. Phys.* **250**, 371 (2004); C. H. Bennett et al., *IEEE Trans. Inform. Theory* **51**, 56 (2005); A. Ambainis, and A. Smith, e-print quant-ph/0404075.
5. W. H. Zurek, *Rev. Mod. Phys.* **75**, 715 (2003).
6. M. Žnidarič, *J. Phys. A* **40**, F105-F111 (2007).
7. K. Życzkowski, and H.-J. Sommers, *J. Phys. A* **34**, 7111 (2001).
8. E. Lubkin, *J. Math. Phys.* **19**, 1028 (1978); D. Page, *Phys. Rev. Lett.* **71**, 1291 (1993); J. Sánchez-Ruiz, *Phys. Rev. E* **52**, 5653-5655 (1995); S. Sen, *Phys. Rev. Lett.* **77**, 1 (1996); V. Cappellini, H.-J. Sommers, and K. Życzkowski, *Phys. Rev. A* **74**, 062322 (2006); P. Hayden, D. W. Leung, and A. Winter, *Commun. Math. Phys.* **265**, 95 (2006); O. Giraud, *J. Phys. A* **40**, 2793-280 (2007); O. Giraud, *J. Phys. A* **40**, F1053-F1062 (2007); L. Viola, and W. G. Brown, *J. Phys. A* **40** 8109-8125 (2007).
9. S. Lloyd, and H. Pagels, *Ann. Phys. (NY)* **188**, 186-213 (1988).
10. V. A. Marčenko, and L. A. Pastur, *Math. USSR-Sb* **1**, 457-483 (1967).
11. S. N. Majumdar, O. Bohigas, and A. Lakshminarayan, *J. Stat. Phys.* **131**, 33 (2008).
12. H. Kubotani, S. Adachi, and M. Toda, *Phys. Rev. Lett.* **100**, 240501 (2008).
13. J. Emerson et al., *Science* **302**, 2098 (2003); Y. S. Weinstein, and C. S. Hellberg, *Phys. Rev. Lett.* **95**, 030501 (2005); L. Arnaud, and D. Braun, *Phys. Rev. A* **75**, 062314 (2007); Y. Most et al., *Phys. Rev. A* **76**, 022328 (2007); W. G. Brown et al., *Phys. Rev. A* **77**, 040303(R) (2008).
14. J. Emerson, E. Livine, and S. Lloyd, *Phys. Rev. A* **72**, 060302(R) (2005);
15. R. Oliveira, O. C. O. Dahlsten, and M. B. Plenio, *Phys. Rev. Lett.* **98**, 130502 (2007); O. C. O. Dahlsten, R. Oliveira, and M. B. Plenio, *J. Phys. A* **40**, 8081 (2007).
16. M. Žnidarič, *Phys. Rev. A* **76**, 012318 (2007).
17. N. Khaneja, R. Brockett, and S. J. Glaser, *Phys. Rev. A* **63**, 032308 (2001).
18. A. Harrow, and R. Low, e-print arXiv:0802.1919.
19. M. Žnidarič, e-print arXiv:0809.0554, *Phys. Rev. A., in press.*
20. J. L. Lebowitz, *Physica A* **263**, 516 (1999).
21. M. Žnidarič, T. Prosen, G. Benenti, and G. Casati, *J. Phys. A* **40**, 13787 (2007).
22. M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Lett. A* **223**, 1 (1996).
23. M. Žnidarič, *Phys. Rev. A* **78**, 022105 (2008).
24. V. M. Kendon, K. Nemoto, and W. J. Munro, *J. Mod. Opt.* **49**, 1709 (2002); V. M. Kendon, K. Życzkowski, and W. J. Munro, *Phys. Rev. A* **66**, 062310 (2002).
25. M. L. Mehta, *Random Matrices*, 2nd. ed. Academic Press, 1991.