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Decoherence of spin echoes

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Abstract

We define a quantity, the so-called *purity fidelity*, which measures the rate of dynamical irreversibility due to decoherence, observed e.g. in *echo* experiments, in the presence of an arbitrary small perturbation of the total (system + environment) Hamiltonian. We derive a linear response formula for the purity fidelity in terms of integrated time correlation functions of the perturbation. Our relation predicts, similar to the case of fidelity decay, that the faster the decay of purity fidelity the slower is the decay of time correlations. In particular, we find exponential decay in quantum mixing regime and faster, initially quadratic and later typically Gaussian decay in the regime of non-ergodic, e.g. integrable quantum dynamics. We illustrate our approach by an analytical calculation and numerical experiments in the Ising spin 1/2 chain kicked with tilted homogeneous magnetic field where part of the chain is interpreted as a system under observation and part as an environment.

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1. Introduction

The relation between the rate of decoherence and the nature of dynamics is one of the central issues in quantum mechanics of non-integrable systems. Zurek proposed [1–3] several quite intuitive results bridging between the mechanisms of classical chaotic dynamics and quantum interference. In particular, it has been shown [3] that von Neumann entropy of the *reduced* (environment averaged) density matrix of an open quantum system (system + environment) grows (on a short timescale) with the rate given by the Lyapunov exponents of the corresponding classical dynamics.

However, this is expected to be true only under two rather severe conditions: (1) the system has to be initially prepared in a 'coherent' state (i.e. minimum uncertainty wave packet) and (2) one has to be deep in the semi-classical regime of very small effective Planck constant

$\hbar$ since the timescale of expected quantum–classical correspondence of the entropy growth, namely the *Ehrenfest time*, scales as $t_{\rm E} \propto \log(1/\hbar)$. Indeed, performing real or even numerical experiments within this regime appears to be very difficult, see e.g. the numerical experiment in an N-atom Jaynes–Cummings model [4], where a difference of the entropy growth between integrable and classically chaotic cases has been observed, but its qualitative nature does not become clear.

On the other hand, by rejecting assumption (1) on the coherent initial state but considering a random pure initial state instead, one can use orthogonal/unitary invariance and define simple random matrix models to analyse the time evolution of decoherence [5], and no qualitative and minimal quantitative differences between regular and chaotic dynamics were observed. Furthermore, we should stress that in quantum information science considering random initial states is potentially much more useful. Indeed, in order to use a massive parallelism of quantum computation one has to prepare the quantum computer system initially in a coherent superposition (pure state) of as many elementary qubit (basis) states as possible, but to contain maximal information this state can and should be any possible state and thus will behave more like a random state than like a Gaussian wave packet.

Based on a proposition by Peres [6], there has been much interest in viewing reversibility of a process with an imperfection on the reversed time evolution rather than on the state as discussed by Casati *et al* [7]. This allows us to avoid some of the essential implications of linearity of quantum mechanics that trivialize the latter case. The related *state correlation function* is usually called *fidelity*. Its behaviour for chaotic systems on the Ehrenfest timescale has first been discussed in [8] with similar findings as in [1–3]. On the other hand, it has been found recently [9–11] that on more relevant timescales related to the decay of correlation functions of quantum observables in the mixing case the fidelity decay is indeed exponential, but with a very different exponent, which is determined basically by the strength of the perturbation. This result was obtained for any state and, what is more, for the integrable case a faster Gaussian decay was found in the thermodynamic limit [9]. More strictly speaking, the result implied for *small times* or *small perturbation strengths* linear decay behaviour in the mixing case and quadratic decay in the integrable or more generally non-ergodic case, where 'small times' may still be large compared to the perturbative regime where all dependences are quadratic anyway. It should be stressed that the fidelity can essentially be interpreted as an autocorrelation function in the interaction picture and the universal initial quadratic decay in the perturbative regime is known as the *quantum Zeno effect*. However, in the present paper and in the previous works [9–11] we are interested in longer timescales, much beyond the Zeno regime, where any surviving quadratic decay has its dynamical origin.

Decoherence often follows correlation functions and in the present situation decoherence should behave along the same lines as fidelity. Yet it is worthwhile to test this, as understanding decoherence is essential for the quantum information applications. The central issue of this paper is thus to test the evolution of decoherence of echoes (some aspects of which are relevant in the real spin–echo experiments [12]). Indeed it is crucial to determine if fidelity is a reliable measure. This is all the more true because of the counter-intuitive result that fidelity is higher for mixing systems than for integrable ones. Due to the great interest in quantum information problems we shall mainly focus on random states for the reason pointed out above.

Basically, we follow previous work on fidelity [9–11] and express our results on decoherence by time correlation functions of quantum observables, though we shall have to generalize the concept of correlation function slightly. We shall use the purity [2] (which is one minus the linear entropy or idempotency defect used in [4]) instead of the von Neumann entropy as a measure of decoherence, because purity, being an analytic function of the density matrix, is much easier to handle. Rather than introducing non-unitary time evolution we follow

the simpler way used in [4, 5] of partial tracing over the environment in a product Hilbert space after unitary time evolution. The kicked spin chain will serve as a specific example for our considerations, where analytic calculations for the integrable case can be carried further and numerics will confirm the validity of our approximations.

The unitary propagator U of the total system (central system + environment) can be either a short time-propagator $U = \exp(-\mathrm{i}H\Delta t/\hbar)$, a Floquet map $U = \hat{\mathcal{T}}\exp\left(-\mathrm{i}\int_0^p \mathrm{d}\tau\, H(\tau)/\hbar\right)$ for a periodically time-dependent Hamiltonian H ($H(\tau + p) = H(\tau)$) or any abstract quantum map. We only assume that the total Hilbert space $\mathcal{H}$ (the domain of U) can be written as a direct product of two parts,

$$\mathcal{H} = \mathcal{H}_\mathrm{c} \otimes \mathcal{H}_\mathrm{e}. \tag{1}$$

In what follows, the subscripts 'c' and 'e' will denote quantities referring to the *central system* and the *environment*, respectively, and to the total system when no subscript is attached.

As mentioned above we shall not introduce a non-unitary time evolution, but just consider the decoherence induced in the central system by its entanglement with the environment and then perform partial traces over the latter. In other words, we shall test the stability of the disentanglement caused by the inverse time evolution operator if this operator is perturbed. This perturbation can generally be described by some self-adjoint operator A over $\mathcal{H}$ and some perturbation strength δ to yield

$$U_\delta = U\exp(-\mathrm{i}A\delta/\hbar). \tag{2}$$

Our perturbation is static (the generator A has no explicit time dependence) and cannot be associated with noise since it is not due to the coupling to the environment (which is explicitly included from the outset) but stems from the imperfect (or unknown) description of the total Hamiltonian.

First, let us consider coherent quantum evolution of the total system, and define the *fidelity* as an overlap between states evolving under unperturbed and perturbed time evolution,

$$F(t) = \langle\psi_\delta(t)|\psi(t)\rangle = \langle\psi|U_\delta^{-t}U^t|\psi\rangle \tag{3}$$

where our time t is an integer. Fidelity has become a standard measure of instability of quantum computation [13], but has also been considered in a more abstract context as a measure for the instability of quantum dynamics [6, 8–11, 14–19]. It has recently been pointed out [9–11] that fidelity is intimately related to the decay of correlations, and that, surprisingly enough, the faster the fidelity decay the slower is the decay of correlations. However, fidelity is a property of a pure state of the total system which is typically not accessible. What can *de facto* be measured is only the information relating to the reduced density operator

$$\rho_\mathrm{c}(t) = \mathrm{Tr}_\mathrm{e}|\psi(t)\rangle\langle\psi(t)| \tag{4}$$

where Tr_e is a *partial trace* over the environmental degrees of freedom $\mathcal{H}_\mathrm{e}$.

We now have to obtain an extension of the concept of fidelity in order to measure coherence properties of the reduced density matrix. Assume we prepare our system in a pure product (disentangled) state

$$|\psi\rangle = |\psi_\mathrm{c}\rangle \otimes |\psi_\mathrm{e}\rangle \tag{5}$$

such that the reduced density matrix is also pure $\rho_\mathrm{c}(0) = |\psi_\mathrm{c}\rangle\langle\psi_\mathrm{c}|$. Then we propagate our system for some time t, after which the state $U^t|\psi\rangle$ becomes generally an entangled superposition of system and environment states. We then invert time, i.e. we change the sign of the Hamiltonian with a small inaccuracy described by the operator A (2), and propagate the system backwards for the same amount of time arriving at the final 'echo' state,

$$|\phi(t)\rangle = U_\delta^{-t}U^t|\psi\rangle. \tag{6}$$

The overlap between the initial and the final state would be just a fidelity (3), $F(t) = \langle\phi(t)|\psi\rangle$, however here we are interested in only to what extent has the final state disentangled from the environment, i.e. how much the final reduced density operator

$$\rho_{\rm c}^{\rm echo}(t) = {\rm Tr}_{\rm e}|\phi(t)\rangle\langle\phi(t)| \tag{7}$$

deviates from a pure state. This is best quantified in terms of a *purity*, ${\rm Tr}\,\rho^2$, which is equal to 1 for a pure state and less than 1 otherwise. The minimal value of purity is $1/\mathcal{N}_{\rm c}$ where $\mathcal{N}_{\rm c}$ is the dimension of $\mathcal{H}_{\rm c}$; note that this limiting value can only be reached if the dimension of the environment $\mathcal{N}_{\rm e}$ tends to infinity. For finite-dimensional environment space, we find later a more accurate limiting value (34) (see also [5]). We shall therefore study *the purity fidelity* defined as the purity of an echoed reduced state

$$F_{\rm P}(t) = {\rm Tr}_{\rm c}\left[\rho_{\rm c}^{\rm echo}(t)\right]^2 = {\rm Tr}_{\rm c}\left[{\rm Tr}_{\rm e}\left(U_\delta^{-t}U^t|\psi_{\rm c}\rangle|\psi_{\rm e}\rangle\langle\psi_{\rm c}|\langle\psi_{\rm e}|U^{-t}U_\delta^t\right)\right]^2. \tag{8}$$

In section 2 we shall make our main theoretical predictions on purity fidelity in relation to ergodic properties of dynamics and in particular to the correlation decay. In section 3 we shall apply our results in the quantum spin chain model, namely the Ising spin 1/2 chain kicked with a tilted homogeneous magnetic field, which exhibits all qualitatively different regimes of quantum dynamics ranging from integrable to mixing. In section 4 we shall discuss some implications of our results and conclude.

2. The relation between purity fidelity and correlation decay

2.1. Linear response

Following [9–11] we start from time-dependent perturbation theory (linear response) and expand the purity fidelity in a power series in the perturbation strength. It is convenient to write the complete basis of $\mathcal{H}$ as $|j,\nu\rangle$, where Latin/Greek indices running over $\mathcal{N}_{\rm c,e} = \dim\mathcal{H}_{\rm c,e}$ values denote system/environmental degrees of freedom, such that the initial state is always designated as $|1,1\rangle = |\psi_{\rm c}\rangle\otimes|\psi_{\rm e}\rangle$. First, let us rewrite the purity fidelity,

$$F_{\rm P}(t) = {\rm Tr}_{\rm c}[{\rm Tr}_{\rm e}(M_t|\psi_{\rm c}\rangle|\psi_{\rm e}\rangle\langle\psi_{\rm c}|\langle\psi_{\rm e}|M_t^\dagger)]^2 \tag{9}$$

$$= \sum_{j,k,\mu,\nu}\langle j,\mu|M_t|1,1\rangle\langle 1,1|M_t^\dagger|k,\mu\rangle\langle k,\nu|M_t|1,1\rangle\langle 1,1|M_t^\dagger|j,\nu\rangle \tag{10}$$

in terms of a *unitary fidelity operator*

$$M_t = U_\delta^{-t}U^t. \tag{11}$$

Second, we observe [9] that the fidelity operator can be rewritten in terms of the perturbing operator in the Heisenberg picture $A_t = U^{-t}AU^t$, namely

$$M_t = \exp({\rm i}A_0\delta/\hbar)\exp({\rm i}A_1\delta/\hbar)\cdots\exp({\rm i}A_{t-1}\delta/\hbar). \tag{12}$$

This expression can be formally expanded into a power series,

$$M_t = 1 + \sum_{m=1}^{\infty}\frac{{\rm i}^m\delta^m}{m!\hbar^m}\hat{\mathcal{T}}\sum_{t_1,\ldots,t_m=1}^{t}A_{t_1}A_{t_2}\cdots A_{t_m} \tag{13}$$

which always converges (in the *strong limit* sense) provided the generator A is a bounded operator. The symbol $\hat{\mathcal{T}}$ denotes a left-to-right time ordering of operator products. Third, we truncate expression (13) at the second order of δ^2 and plug it into expression (9). After a tedious but straightforward calculation, we find

$$F_{\rm P}(t) = 1 - \frac{2\delta^2}{\hbar^2}\sum_{t',t''=0}^{t-1}\sum_{j}^{j\neq 1}\sum_{\nu}^{\nu\neq 1}\langle 1,1|A_{t'}|j,\nu\rangle\langle j,\nu|A_{t''}|1,1\rangle + \mathcal{O}(\delta^4). \tag{14}$$

The RHS of this linear response formula indeed looks much like a time correlation function, however with funny exclusion rules on the state summation. It can be written more elegantly in a basis independent way as

$$F_{\mathrm{P}}(t) = 1 - \frac{2\delta^2}{\hbar^2}\,\mathrm{Tr}\Big[\rho \sum A_t \tilde{\rho} \sum A_t\Big] + \mathcal{O}(\delta^4) \tag{15}$$

where $\sum A_t \equiv \sum_{t'=0}^{t-1} A_{t'}$, $\rho = |\psi\rangle\langle\psi|$, and $\tilde{\rho} = (1_{\mathrm{c}} - |\psi_{\mathrm{c}}\rangle\langle\psi_{\mathrm{c}}|) \otimes (1_{\mathrm{e}} - |\psi_{\mathrm{e}}\rangle\langle\psi_{\mathrm{e}}|)$. Our expectation is that the properties of purity fidelity decay will mainly depend on dynamics, i.e. the behaviour of the operator $\sum A_t$, and less on the detailed structure of the initial state encoded in ρ and $\tilde{\rho}$. Let us first discuss the limiting qualitatively different cases of dynamics.

2.1.1. Regime of ergodicity and mixing. In the regime of *ergodic and mixing* quantum dynamics [20], the *reduced transport coefficient*

$$\tilde{\sigma} = \lim_{t\to\infty} \frac{1}{2t}\,\mathrm{Tr}\Big[\rho \sum A_t \tilde{\rho} \sum A_t\Big] \tag{16}$$

can be estimated in terms of the usual Kubo transport coefficient

$$\sigma = \lim_{t\to\infty} \frac{1}{2t} \sum_{t',t''=0}^{t-1} \mathrm{Tr}[\rho A_{t'} A_{t''}] \tag{17}$$

which is obviously finite $\sigma < \infty$ in the case of mixing dynamics and sufficiently strong decay of time correlations. Namely, one can easily prove that $0 \leqslant \tilde{\sigma} \leqslant \sigma$ since all the terms of σ missing in $\tilde{\sigma}$, as well as all the terms of $\tilde{\sigma}$ when expanding along (14), are non-negative. For times t larger than a certain *mixing time* t_{mix}, $t > t_{\mathrm{mix}}$, i.e. a characteristic timescale on which the limiting process (16) converges, we thus find a linear decay of purity fidelity

$$F_{\mathrm{P}}(t) = 1 - \frac{4\delta^2}{\hbar^2}\tilde{\sigma} t + \mathcal{O}(\delta^4) = 1 - \frac{4t}{\tilde{\tau}_{\mathrm{em}}} + \mathcal{O}(\delta^4) \tag{18}$$

on a timescale

$$\tilde{\tau}_{\mathrm{em}} = \frac{\hbar^2}{\delta^2 \tilde{\sigma}} \propto \delta^{-2}. \tag{19}$$

2.1.2. Non-ergodic regime. In the opposite regime of *non-ergodic* (e.g. integrable) quantum dynamics, the non-trivial (e.g. different from a multiple of identity) time-averaged operator exists,

$$\bar{A} = \lim_{t\to\infty} \frac{1}{t} \sum A_t \tag{20}$$

so that for times larger than a certain *averaging time* $t > t_{\mathrm{ave}}$, i.e. a characteristic timescale on which the limiting process (20) converges, we find a quadratic decay of purity fidelity

$$F_{\mathrm{P}}(t) = 1 - \frac{2\delta^2}{\hbar^2}\tilde{D} t^2 + \mathcal{O}(\delta^4) = 1 - 2\left(\frac{t}{\tilde{\tau}_{\mathrm{ne}}}\right)^2 + \mathcal{O}(\delta^4) \tag{21}$$

with a coefficient which we name as *reduced stiffness*

$$\tilde{D} = \mathrm{Tr}[\rho \bar{A} \tilde{\rho} \bar{A}] \tag{22}$$

on a timescale

$$\tilde{\tau}_{\mathrm{ne}} = \frac{\hbar}{\delta\sqrt{\tilde{D}}} \propto \delta^{-1}. \tag{23}$$

We note that our reduced stiffness coefficient is again rigorously bounded by the common (non-reduced) stiffness

$$D = \mathrm{Tr}[\rho \bar{A}^2] - \mathrm{Tr}[\rho \bar{A}]^2 \tag{24}$$

namely $\tilde{D} \leqslant D$.

It should be stressed that this regime of non-ergodic dynamics covers not only the integrable cases but also the more general intermediate cases of (classically) mixed systems. Namely, in a generic mixed system one should expect that typical observables would have non-trivial time-averages $\bar{A}$, and so typically $D \neq 0$, $\tilde{D} \neq 0$ unlike in an ergodic system where $D = \tilde{D} = 0$ for all suitable observables A.

2.2. Beyond the linear response

Next we extend our linear-response results to the regime, where the value of purity fidelity becomes considerably lower than 1. Again we consider the two qualitatively different cases of dynamics.

2.2.1. Regime of ergodicity and mixing. Here we assume, in full analogy with the derivation of fidelity decay [9, 11], that also the property of quantum n-mixing holds (in the asymptotics $\mathcal{N} \to \infty$), namely that all the higher n-point time correlation functions factorize so that the δ-expansion of the fidelity operator can be summed up and in the *weak limit* sense we obtain

$$\mathrm{e}^{t/\tau_{\mathrm{em}}} M_t \to 1 \qquad \text{as} \quad t \gg t_{\mathrm{mix}} \quad \text{where} \quad \tau_{\mathrm{em}} = \frac{\hbar^2}{\delta^2 \sigma}. \tag{25}$$

This statement is equivalent to the statement shown in [9, 11] that for arbitrary (pure or mixed) state ρ, fidelity decay $F(t \gg t_{\mathrm{mix}}) = \mathrm{Tr}\,\rho M_t = \exp(-t/\tau_{\mathrm{em}})$ is independent of the state. In the case of a semi-classical situation of small effective value of $\hbar$ the limit (25) starts to build up [11] only after the Ehrenfest time t_{E}, $t_{\mathrm{E}} = \ln(1/\hbar)/\lambda$, where λ is the largest classical Lyapunov exponent, so then $t_{\mathrm{mix}} = t_{\mathrm{E}}$. Of course, (25) never holds in the strong limit sense as the fidelity operator M_t is unitary. Plugging (25) into formula (9) we obtain for the asymptotic exponential decay of purity fidelity

$$F_{\mathrm{P}}(t) = \exp(-4t/\tau_{\mathrm{em}}) \qquad \text{as} \quad t \gg t_{\mathrm{mix}}. \tag{26}$$

Comparing equations (26) and (18) in the asymptotic regime $t \to \infty$ while approaching $\delta \to 0$ so as to remain in the regime of linear response, we find that we must have strictly $\tilde{\tau}_{\mathrm{em}} = \tau_{\mathrm{em}}$, i.e.

$$\tilde{\sigma} = \sigma. \tag{27}$$

Of course, this argument is correct only for sufficiently strong mixing, e.g. for exponential decay of correlations. For weaker forms of mixing, e.g. for *power-law* decay of correlations, one may not strictly justify equation (25) by summing up equation (13) to all orders, though it may still be correct in some cases, e.g. due to sufficiently high power of decay or due to alternating signs in correlation functions. In other cases, one may find anomalous non-exponential decay of the fidelity or the fidelity operator.

2.2.2. Non-ergodic regime. Again, in analogy to the simple fidelity decay in non-ergodic situations [9, 11], we can rewrite the fidelity operator in terms of the time-averaged operator $\bar{A}$,

$$M_t \to \exp(\mathrm{i}\bar{A}t\delta/\hbar) \qquad \text{as} \quad t \gg t_{\mathrm{ave}}. \tag{28}$$

Plugging the fidelity operator (28) into formula (9) we find that, as a manifestation of non-ergodicity, the behaviour of purity fidelity generally depends on the structure of the initial state.

2.3. State-averaged purity fidelity

As we stated in the introduction, we are interested in the behaviour of a random initial state of the central system and as far as the environment is concerned we as well average over states to reflect our ignorance of the latter. Let us therefore start with a *random product initial state* (5) implying that the states $|\psi_c\rangle$ and $|\psi_e\rangle$ are random $\mathcal{N}_c$ and $\mathcal{N}_e$ dimensional vectors, whose components $\langle j|\psi_c\rangle$ and $\langle\mu|\psi_e\rangle$ are, in the limits $\mathcal{N}_c \to \infty$ and $\mathcal{N}_e \to \infty$, *independent complex random Gaussian variables* with variance $1/\mathcal{N}_c$ and $1/\mathcal{N}_e$, respectively. Denoting averaging over random product initial states as $\langle\cdot\rangle_\psi$ one can easily average the linear response formula (14),

$$\langle F_P(t)\rangle_\psi = 1 - \frac{2\delta^2\mathcal{N}_c\mathcal{N}_e}{\hbar^2(\mathcal{N}_c+1)(\mathcal{N}_e+1)} \sum_{t',t''=0}^{t-1} C_P(t',t'') + \mathcal{O}(\delta^4) \tag{29}$$

expressing the purity fidelity decay in terms of a sum of *reduced correlation function*

$$C_P(t',t'') = \langle A_{t'}A_{t''}\rangle + \langle A\rangle^2 - \langle\langle A_{t'}\rangle_e\langle A_{t''}\rangle_e\rangle_c - \langle\langle A_{t'}\rangle_c\langle A_{t''}\rangle_c\rangle_e \tag{30}$$

where $\langle\cdot\rangle_c \equiv (1/\mathcal{N}_c)\mathrm{Tr}_c(\cdot)$, $\langle\cdot\rangle_e \equiv (1/\mathcal{N}_e)\mathrm{Tr}_e(\cdot)$ and $\langle\cdot\rangle \equiv \langle\langle\cdot\rangle_e\rangle_c = (1/\mathcal{N})\,\mathrm{Tr}(\cdot)$.

Again, in the ergodic and mixing regime, where correlation functions (30) decay fast as $|t''-t'| \to \infty$, the average purity fidelity exhibits initial linear decay (18) with average transport coefficient

$$\langle\tilde{\sigma}\rangle_\psi = \lim_{t\to\infty}\frac{1}{2t}\sum_{t',t''=0}^{t-1} C_P(t',t''). \tag{31}$$

Since, for sufficiently strong mixing we have (27), $\langle\sigma\rangle_\psi = \lim_{t\to\infty}\frac{1}{2t}\sum_{t',t''=0}^{t-1} C(t'-t'') = \langle\tilde{\sigma}\rangle_\psi$, where $C(t) = \langle A_t A\rangle - \langle A\rangle^2$, then $\lim_{t\to\infty}\frac{1}{t}\sum_{t',t''=0}^{t-1}(C(t'-t'') - C_P(t',t'')) = 0$. We note again that all the results referring to the ergodic and mixing regime implicitly assume the limits $\mathcal{N}_c, \mathcal{N}_e \to \infty$ to be considered before $t \to \infty$.

On the other hand, in the non-ergodic regime we have quadratic initial decay (21) with state-averaged reduced stiffness

$$\langle\tilde{D}\rangle_\psi = \frac{\mathcal{N}_c\mathcal{N}_e}{(\mathcal{N}_c+1)(\mathcal{N}_e+1)}\left(\langle\bar{A}^2\rangle + \langle\bar{A}\rangle^2 - \left\langle\langle\bar{A}\rangle_c^2\right\rangle_e - \left\langle\langle\bar{A}\rangle_e^2\right\rangle_c\right). \tag{32}$$

Without making reference to any of the two extreme cases, the general expression for the purity fidelity (10) may be simply state averaged while keeping the fidelity operator completely general:

$$\begin{aligned}\langle F_P(t)\rangle_\psi = {} & \frac{\mathcal{N}_c+\mathcal{N}_e}{(\mathcal{N}_c+1)(\mathcal{N}_e+1)} + \frac{1}{\mathcal{N}_e(\mathcal{N}_e+1)\mathcal{N}_c(\mathcal{N}_c+1)} \\ & \times \sum_{j,k,p,q,\alpha,\beta,\mu,\nu} \big(\langle p,\alpha|M_t|j,\nu\rangle\langle j,\mu|M_t^\dagger|q,\alpha\rangle\langle q,\beta|M_t|k,\mu\rangle\langle k,\nu|M_t^\dagger|p,\beta\rangle \\ & + \langle p,\alpha|M_t|j,\nu\rangle\langle k,\nu|M_t^\dagger|q,\alpha\rangle\langle q,\beta|M_t|k,\mu\rangle\langle j,\mu|M_t^\dagger|p,\beta\rangle\big).\end{aligned} \tag{33}$$

From (33) one observes that (averaged) purity fidelity does not decay to zero. It has a finite saturation plateau value F^*, which can be estimated by assuming that after a long time t and for

sufficiently strong perturbation δ (such that the eigenstates of U_δ look random in the basis of eigenstates of U) the fidelity operator M_t becomes a matrix of independent Gaussian random variables. Then the average of the product of four matrix elements (33) can be estimated by a pair-contraction rule with $\langle\langle p,\alpha|M_t|j,\nu\rangle\langle q,\beta|M_t^\dagger|k,\mu\rangle\rangle = \frac{1}{\mathcal{N}_e\mathcal{N}_c}\delta_{pk}\delta_{\alpha\mu}\delta_{qj}\delta_{\beta\nu}$ as

$$F^* = \frac{\mathcal{N}_c + \mathcal{N}_e + 4 + \mathcal{O}(1/\mathcal{N}_c) + \mathcal{O}(1/\mathcal{N}_e)}{(\mathcal{N}_c+1)(\mathcal{N}_e+1)}. \tag{34}$$

We have defined a reasonable measure for decoherence for the echo process and have determined that purity decays faster for a near integrable system than for a mixing one. This fact is quite surprising in view of previous results on decoherence, but not so much if we consider that it follows closely the results obtained for fidelity. We will proceed to apply our findings to a kicked Ising spin chain.

3. Kicked Ising chain

We shall now consider the application of purity fidelity decay in a class of model systems, namely one-dimensional spin 1/2 chains. We consider a chain of L spins described by Pauli operators σ_j^α, $j \in \{0,1,\ldots,L-1\}$, $\alpha \in \{x,y,z\}$, with periodic boundary conditions $\sigma_{j+L}^\alpha \equiv \sigma_j^\alpha$. In particular, we concentrate on the example of the kicked Ising (KI) model [9] with the Hamiltonian

$$H_{\mathrm{KI}}(t) = \sum_{j=0}^{L-1}\left\{J_z\sigma_j^z\sigma_{j+1}^z + \delta_p(t)\left(h_x\sigma_j^x + h_z\sigma_j^z\right)\right\} \tag{35}$$

where $\delta_{\mathrm{p}}(t) = \sum_{m=-\infty}^{\infty}\delta(t-mp)$ is a periodic delta function, generating the Floquet-map

$$U = \exp\left(-\mathrm{i}J_z\sum_j \sigma_j^z\sigma_{j+1}^z\right)\exp\left(-\mathrm{i}\sum_j\left(h_x\sigma_j^x + h_z\sigma_j^z\right)\right) \tag{36}$$

where we take units such that $p = \hbar = 1$, depending on a triple of independent parameters (J_z, h_x, h_z). KI is *completely integrable* for longitudinal ($h_x = 0$) and transverse ($h_z = 0$) fields [21], and has finite parameter regions of ergodic and non-ergodic behaviour in the thermodynamic limit $L \to \infty$ for a tilted field (see figure 1). The non-trivial integrability of a transverse kicking field, which somehow inherits the solvable dynamics of its well-known autonomous version [22], is quite remarkable since it was shown [21] that the Heisenberg dynamics can be calculated explicitly for observables which are bilinear in Fermi operators $c_j = \left(\sigma_j^y - \mathrm{i}\sigma_j^z\right)\prod_{j'}^{j'<j}\sigma_{j'}^x$ with time correlations $C(t)$ decaying to the non-ergodic stationary values D as $|C(t)-D| \sim t^{-3/2}$. In order to test our predictions of purity fidelity decay by explicit calculation and/or numerical experiment, we consider a line in three-dimensional parameter space, with fixed $J_z = 1$, $h_x = 1.4$ and varying h_z exhibiting all different types of dynamics: (*a*) *integrable* for $h_z = 0$, (*b*) *intermediate* (non-integrable and non-ergodic) for $h_z = 0.4$ and (*c*) *ergodic and mixing* for $h_z = 1.4$. In all cases, we fix the operator $A = M := \sum_j \sigma_j^x$ which generates the following parametric perturbation of the KI model with

$$\begin{aligned} h_x &\to h_x + \frac{\left(h_x^2 + h_z^2 h\cot h\right)}{h^2}\delta + \mathcal{O}(\delta^2) \\ h_z &\to h_z + \frac{h_x h_z(1 - h\cot h)}{h^2}\delta + \mathcal{O}(\delta^2) \qquad \text{where} \quad h = \sqrt{h_x^2+h_z^2} \end{aligned} \tag{37}$$

and vary the size L and the perturbation strength δ.

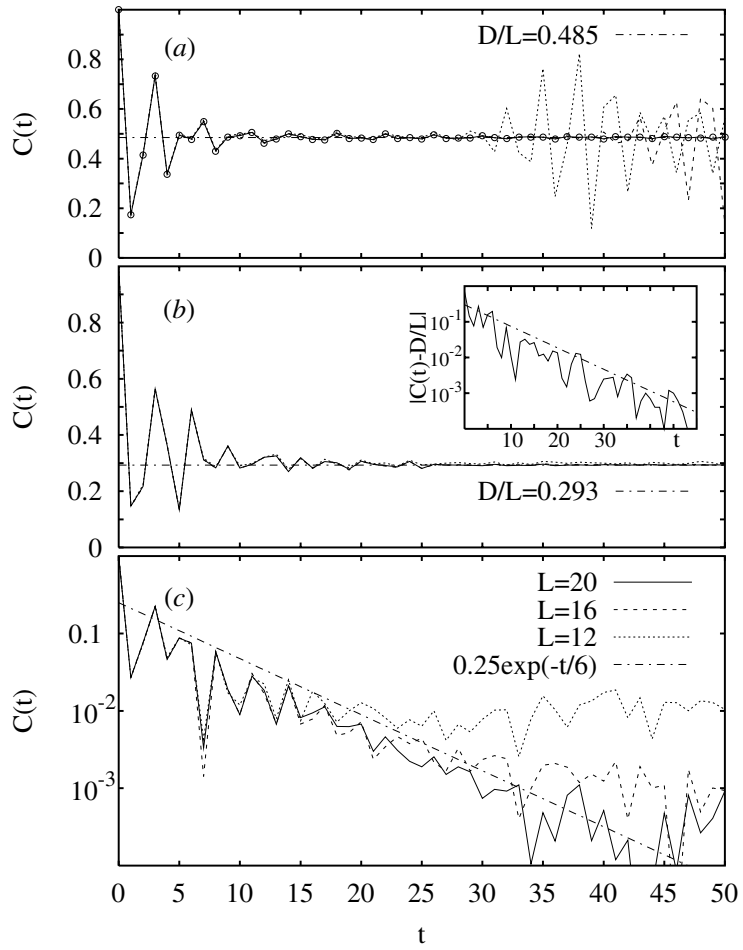

Figure 1. Correlation decay for three cases of KI: (*a*) integrable $h_z = 0$, (*b*) intermediate $h_z = 0.4$ and (*c*) ergodic $h_z = 1.4$, for different sizes $L = 20, 16, 12$ (solid–dotted connected curves, almost indistinguishable in (*a*) and (*b*)). Circles (*a*) show the exact $L = \infty$ result. Chain lines are theoretical/suggested asymptotics (see text).

For the sake of mathematical simplicity of the model, we do not want to introduce extra degrees of freedom to simulate the environment. Instead, we logically split the spin-chain by assigning a subset $\mathcal{J}_c$ of L_c spins to the central system and the complement $\mathcal{J}_e = \mathbb{Z}_L \backslash \mathcal{J}_c$ of $L_e = L - L_c$ spins to the environment. Then we have $\mathcal{N}_c = 2^{L_c}, \mathcal{N}_e = 2^{L_e}, \mathcal{N} = 2^L$. In our numerical experiments and explicit analytical calculations, we shall consider three special cases (in the first two cases we have to assume that L is even):

- (A) *Alternating subchains*, where every second spin is assigned to the central system $\mathcal{J}_c^A = \{0, 2, 4, \ldots, L-2\}$. This situation is supposed to be a good model of a situation where all central system's degrees of freedom are directly coupled to the environment.
- (C) *Connected subchains*, where a connected half of the chain is assigned to the central system $\mathcal{J}_c^C = \{0, 1, 2, \ldots, L/2 - 1\}$. Here the central system is coupled to the environment just at two ending points, namely $j = 0$ and $j = L/2 - 1$.

- (S) *Single spin*, where the central system consists of a single spin $\mathcal{J}_c^S = \{0\}$, i.e. a two-level quantum system coupled to a correlated many-body environment.

Our explicit calculations will be performed for a random initial state, in other words only state-averaged purity fidelity will be computed.

3.1. Exact calculation for the integrable case

Here we show how to extend our analytical approach [21] to solve purity fidelity decay in the integrable case $h_z = 0$. We start by a finite size version of the formalism, however the final results will be most elegant and simple in the thermodynamic limit $L \to \infty$. The main quantity which we want to calculate is a state-averaged reduced stiffness $\langle \tilde{D} \rangle_\psi$ (32). As a first step in our calculation, we have to construct a time-averaged operator $\bar{A}$ associated with the perturbing operator A. This can be done, as described in [21], by means of the invariant space of the *adjoint* map U^{ad}, which is a unitary operator defined on a linear space of (bounded) observables as $U^{\mathrm{ad}}A = U^\dagger A U$. The complete basis of this invariant space can be constructed in terms of two sequences of $2L$ operators, U_n, V_n, $n \in \mathbb{Z}_{2L}$, namely

$$\begin{aligned} U_n &= \sum_{j=0}^{L-1} \begin{cases} \sigma_j^y \left(\sigma_j^x\right)_{n-1} \sigma_{j+n}^y & n \geqslant 1 \\ -\sigma_j^x & n = 0 \\ \sigma_j^z \left(\sigma_j^x\right)_{-n-1} \sigma_{j-n}^z & n \leqslant -1 \end{cases} \\ V_n &= \sum_{j=0}^{L-1} \begin{cases} \sigma_j^y \left(\sigma_j^x\right)_{n-1} \sigma_{j+n}^z & n \geqslant 1 \\ 1 & n = 0 \\ -\sigma_j^z \left(\sigma_j^x\right)_{-n-1} \sigma_{j-n}^y & n \leqslant -1 \end{cases} \end{aligned} \tag{38}$$

where $\left(\sigma_j^x\right)_k := \prod_{l=1}^k \sigma_{j+l}^x$ for $k \geqslant 1$, $\left(\sigma_j^x\right)_0 := 1$, satisfying a Lie algebra

$$\begin{aligned} &[U_m, U_n] = 2\mathrm{i}(V_{m-n} - V_{n-m}) \\ &[V_m, V_n] = 0 \\ &[U_m, V_n] = 2\mathrm{i}(U_{m+n} - U_{m-n}). \end{aligned} \tag{39}$$

We note the orthogonality

$$\langle U_n U_m \rangle = \langle V_n V_m \rangle = L\delta_{nm} \qquad \langle U_n V_m \rangle = 0. \tag{40}$$

Now, the Floquet-map (36) can be written in terms of the elements of algebra (38) only, namely $U = \exp(-\mathrm{i}JU_1)\exp(\mathrm{i}h_x U_0)$, and the complete basis of an invariant space

$$U^{\mathrm{ad}}\Psi(\varphi) = \Psi(\varphi) \tag{41}$$

can be, for arbitrary finite L, written as $\Psi(\varphi_l)$ with $\varphi_l = \frac{\pi l}{L}$, $l \in \mathbb{Z}_{2L}$, and

$$\begin{aligned} &\Psi(\varphi_l) = \sum_{n=0}^{2L-1} \left(\boldsymbol{u}(\varphi_l)\,\mathrm{e}^{\mathrm{i}n\varphi_l} + \boldsymbol{u}(-\varphi_l)\,\mathrm{e}^{-\mathrm{i}n\varphi_l}\right) \cdot \boldsymbol{E}_n \\ &\boldsymbol{E}_n \equiv \left(U_n, U_{-n}, 2^{-1/2}(V_n - V_{-n})\right) \\ &\boldsymbol{u}(\varphi) \equiv (\cot\alpha - \cot\beta\,\mathrm{e}^{-\mathrm{i}\varphi}, \cot\alpha - \cot\beta\,\mathrm{e}^{\mathrm{i}\varphi}, \sqrt{2}\mathrm{i}\sin\varphi) \end{aligned} \tag{42}$$

where we introduced the angles $\alpha \equiv 2J$, $\beta \equiv 2h_x$. Note that $\Psi(\varphi_{2L-l}) \equiv \Psi(\varphi_l)$ and the basis (of non-equivalent vectors $l = 0, 1, \ldots, L$) is orthogonal,

$$\langle \Psi^\dagger(\varphi_k)\Psi(\varphi_l)\rangle = 4L^2|\boldsymbol{u}(\varphi_k)|^2(\delta_{k,l} + \delta_{k,2L-l}). \tag{43}$$

The time average of an observable can be simply constructed by means of a projection

$$\bar{A} = \sum_{l=0}^{L} \frac{\langle \Psi^\dagger(\varphi_l) A \rangle}{\langle \Psi^\dagger(\varphi_l) \Psi(\varphi_l) \rangle} \Psi(\varphi_l) \tag{44}$$

namely for the magnetization $A = M = U_0$ we have

$$\bar{M} = \frac{1}{4L} \sum_{l=0}^{2L-1} \frac{\operatorname{Re} u_1(\varphi_l)}{|\boldsymbol{u}(\varphi_l)|^2} \Psi(\varphi_l). \tag{45}$$

In the limit $L \to \infty$ the phase variable becomes continuous $\varphi \in [0, 2\pi)$ and the sums (44) and (45) go over to integrals over φ. The first term of (32) is the stiffness $D = \langle \bar{M}^2 \rangle$ which can be calculated explicitly in the limit $L \to \infty$ [21]. Finite size corrections are proven to be smaller than any power in $1/L$, i.e. they have to be *exponentially small*, since the argument of the sum which is approximated by an integral over φ is an L-independent *analytic* function of $\mathrm{e}^{\mathrm{i}\varphi}$,

$$D = L \frac{\max\{|\cos\alpha|, |\cos\beta|\} - \cos^2\beta}{\sin^2\beta} + \mathcal{O}(\exp(-\text{const}\, L)). \tag{46}$$

In order to calculate the remaining terms of the reduced stiffness (32) we have to compute the *reduced inner product* of a pair of operators, namely

$$\langle\langle A \rangle_{\mathcal{J}} \langle B \rangle_{\mathcal{J}} \rangle_{\mathcal{J}'} = 2^{-|\mathcal{J}'|-2|\mathcal{J}|} \operatorname{Tr}_{\mathcal{J}'}(\operatorname{Tr}_{\mathcal{J}} A \operatorname{Tr}_{\mathcal{J}} B) \tag{47}$$

where $\mathcal{J}$ is some subset of $\mathbb{Z}_L$ (either $\mathcal{J}_\mathrm{c}$ or $\mathcal{J}_\mathrm{e}$) and $\mathcal{J}' = \mathbb{Z}_L \backslash \mathcal{J}$. We start with the basis operators U_n, V_n, for which we find by direct inspection of structure (38) that

$$\begin{aligned} &\langle\langle U_m \rangle_{\mathcal{J}} \langle U_n \rangle_{\mathcal{J}} \rangle_{\mathcal{J}'} = \langle\langle V_m \rangle_{\mathcal{J}} \langle V_n \rangle_{\mathcal{J}} \rangle_{\mathcal{J}'} = L f_n \delta_{mn} \\ &\langle\langle U_m \rangle_{\mathcal{J}} \langle V_n \rangle_{\mathcal{J}} \rangle_{\mathcal{J}'} = 0. \end{aligned} \tag{48}$$

Here the *structure function* f_n is defined by the number of *sequential* sets $\mathcal{C}(l_1, l_2) = \{l_1, l_1 + 1, \ldots, l_2\}$ in $\mathbb{Z}_L$ of length $|n| + 1$ which do not intersect the set $\mathcal{J}$, divided by L,

$$f_n = \frac{1}{L} |\{j \in \mathbb{Z}_L; \mathcal{C}(j, j+n) \cap \mathcal{J} = \emptyset\}| \tag{49}$$

and is extended to $\mathbb{Z}_{2L}$ by putting $f_{-n} \equiv f_{2L-n} := f_n$. Using a *form factor* of sorts

$$F(\varphi) = \sum_{n=0}^{2L-1} f_n \mathrm{e}^{\mathrm{i}n\varphi} \tag{50}$$

we can directly write the reduced inner products within the invariant family (42) generalizing (43), namely

$$\langle\langle \Psi(\varphi')^\dagger \rangle_{\mathcal{J}} \langle \Psi(\varphi) \rangle_{\mathcal{J}} \rangle_{\mathcal{J}'} = 2L \operatorname{Re}\big(\boldsymbol{u}(-\varphi') \cdot \boldsymbol{u}(\varphi) F(\varphi - \varphi') + \boldsymbol{u}(\varphi') \cdot \boldsymbol{u}(\varphi) F(\varphi + \varphi')\big). \tag{51}$$

For the calculation of (32), the reduced inner product of a time-averaged observable $\bar{A}$ with itself, $\big\langle \langle \bar{A} \rangle_{\mathcal{J}}^2 \big\rangle_{\mathcal{J}'}$ is of central importance. It can be computed straightforwardly using

expansion (44) and reduced inner products (51). For the magnetization M we obtain

$$G_F := \frac{1}{L}\left\langle \langle \bar{M}\rangle_{\mathcal{J}}^2 \right\rangle_{\mathcal{J}'} = \frac{1}{4L^2}\sum_{k=0}^{2L-1}\sum_{l=0}^{2L-1} \frac{\mathrm{Re}\, u_1(\varphi_k)\mathrm{Re}\, u_1(\varphi_l)\boldsymbol{u}(\varphi_k)^* \cdot \boldsymbol{u}(\varphi_l)}{|\boldsymbol{u}(\varphi_k)|^2|\boldsymbol{u}(\varphi_l)|^2} F(\varphi_l - \varphi_k). \tag{52}$$

The factor L has been taken out in the definition of the coefficient G_F in order to make it conveniently size-independent for the thermodynamic limit. In order to proceed with explicit calculations, we have to derive the form factors for the three different cases (A, C, S). For each of them we have to compute two generally different reduced inner products with the form factors, F_{c} for $\mathcal{J} = \mathcal{J}_{\mathrm{c}}$, $\mathcal{J}' = \mathcal{J}_{\mathrm{e}}$ and F_{e} for $\mathcal{J} = \mathcal{J}_{\mathrm{e}}$, $\mathcal{J}' = \mathcal{J}_{c}$, corresponding respectively to the last two terms of (32):

- (A): Here, due to symmetry both structure functions are identical $f_n^{\mathrm{A}} = \frac{1}{2}\delta_{n,0}$ giving $F_{\mathrm{c}}^{\mathrm{A}}(\varphi) = F_{\mathrm{e}}^{\mathrm{A}}(\varphi) = \frac{1}{2}$.
- (C): Again, we have a symmetry between the central system and the environment, so $f_n^{\mathrm{C}} = \max\left\{\frac{1}{2} - \frac{|n|}{L}, 0\right\}$ and $F_{\mathrm{c}}^{\mathrm{C}}(\varphi) = F_{\mathrm{e}}^{\mathrm{C}}(\varphi) = \pi\delta_{L/2}(\varphi)$ where $\delta_m(\varphi)$ is a fat periodic delta function,

$$\delta_m(\varphi) = \frac{1}{2\pi m}\left(\frac{\sin(m\varphi/2)}{\sin(\varphi/2)}\right)^2 \tag{53}$$

  with the limit $\lim_{m\to\infty}\delta_m(\varphi) = \sum_{k=-\infty}^{\infty}\delta(\varphi - 2\pi k)$.
- (S): Here, we need to consider two different structure functions $f_{\mathrm{c},n}^{\mathrm{S}} = (L - 1 - |n|)/L$, $f_{\mathrm{e},n}^{\mathrm{S}} = \frac{1}{L}\delta_{n,0}$ giving for the form factors $F_{\mathrm{c}}^{\mathrm{S}}(\varphi) = \frac{2\pi(L-1)}{L}\delta_{L-1}(\varphi)$, $F_{\mathrm{e}}^{\mathrm{S}}(\varphi) = \frac{1}{L}$.

Due to linear dependence of G_F on $F(\varphi)$ (52) we need to compute only two types of the reduced inner products besides the non-reduced one (46), namely for a constant form factor $F(\varphi) = 1$ and for a fat-delta form factor $F(\varphi) = \delta_m(\varphi)$. Tedious but straightforward calculation gives (within accuracy $\mathcal{O}(\exp(-\mathrm{const}\, L))$ beyond *all orders*)

$$\begin{aligned} G_1 &= \begin{cases} 2\cot\beta\sin^2(\alpha/2) & \cos\alpha > \cos\beta \quad \cos\alpha + \cos\beta > 0 \\ -\tan(\beta/2) & \cos\alpha < \cos\beta \quad \cos\alpha + \cos\beta > 0 \\ \cot(\beta/2) & \cos\alpha > \cos\beta \quad \cos\alpha + \cos\beta < 0 \\ 2\cot\beta\cos^2(\alpha/2) & \cos\alpha < \cos\beta \quad \cos\alpha + \cos\beta < 0 \end{cases} \\ G_{\delta_m} &= G_\delta - \frac{1}{4\pi m} \qquad G_\delta := \frac{D}{2\pi L}. \end{aligned} \tag{54}$$

The state-averaged purity fidelity in the regime of linear response (29) can be therefore written as

$$\begin{aligned} \langle F_{\mathrm{P}}(t)\rangle_\psi &= 1 - \frac{2\delta^2\mathcal{N}_{\mathrm{c}}\mathcal{N}_{\mathrm{e}}L}{(\mathcal{N}_{\mathrm{c}}+1)(\mathcal{N}_{\mathrm{e}}+1)}\tilde{G}t^2 + \mathcal{O}(\delta^4) \\ \tilde{G} &= \frac{1}{L}\left(\langle \bar{M}^2\rangle - \left\langle \langle \bar{M}\rangle_{\mathcal{J}}^2\right\rangle_{\mathcal{J}'} - \left\langle \langle \bar{M}\rangle_{\mathcal{J}'}^2\right\rangle_{\mathcal{J}}\right) \end{aligned} \tag{55}$$

where the coefficient in the three different cases of interest reads

$$\tilde{G}^{\mathrm{A}} = 2\pi G_\delta - G_1 \qquad \tilde{G}^{\mathrm{C}} = \frac{1}{L} \qquad \tilde{G}^{\mathrm{S}} = \frac{1}{L}\left(2\pi G_\delta - G_1 + \frac{1}{2}\right). \tag{56}$$

It is very interesting to note that only in case (A) where all environmental degrees of freedom are coupled to the central system and vice versa, the resulting expression for purity decay coefficient $\tilde{G}$ is non-vanishing in the thermodynamic limit $L \to \infty$, so the purity fidelity decays qualitatively in the same way as the (non-reduced) fidelity [9], namely the exponent of quadratic decay in time has the same L dependence. In the other two cases (C, S) where the interaction between the central system and the environment is *local* one has a subtle cancellation of leading order terms giving the resulting coefficient which vanishes as $\propto 1/L$ in the thermodynamic limit thus making the purity fidelity to decay still quadratically but on a much longer timescale (by a factor of L) as compared to the timescale of fidelity decay [9]. Of course, our analytical results for the integrable case only give us a timescale of the purity fidelity decay,

$$\tilde{\tau}_{\mathrm{ne}} = \frac{1}{\delta}\sqrt{\frac{(\mathcal{N}_{\mathrm{c}}+1)(\mathcal{N}_{\mathrm{e}}+1)}{\mathcal{N}_{\mathrm{c}}\mathcal{N}_{\mathrm{e}}L\tilde{G}}} \tag{57}$$

but they cannot tell us anything about the global behaviour of $\langle F_{\mathrm{P}}(t)\rangle_{\psi}$ beyond the regime of linear response. These analytical results are clearly confirmed by direct numerical simulations reported in next subsection.

3.2. Numerical calculations in the general case

In the generally non-integrable cases of the kicked Ising model, we calculate the *partial correlation sums*

$$S_{\mathcal{J}}(t) = \frac{1}{L}\left\langle \left\langle \sum A_t \right\rangle_{\mathcal{J}} \left\langle \sum A_t \right\rangle_{\mathcal{J}} \right\rangle_{\mathcal{J}'} \tag{58}$$

and the *total purity correlator*

$$S_{\mathrm{P}}(t) = \frac{1}{L}\sum_{t',t''=0}^{t-1} C_{\mathrm{P}}(t',t'') = S_{\emptyset}(t) + S_{Z_L}(t) - S_{\mathcal{J}}(t) - S_{\mathcal{J}'}(t) \tag{59}$$

by means of numerical simulation. With these we can calculate purity fidelity and compare to the directly numerically simulated purity fidelity decay (and to the analytical calculation in the integrable case). Note that $S_{\emptyset}(t) = \frac{1}{L}\langle[\sum A_t]^2\rangle = \frac{1}{L}\sum_{t',t''=0}^{t-1}\langle AA_{t'-t''}\rangle$ is the usual (non-reduced) integrated correlation function and $S_{\mathbb{Z}_L}(t) \equiv 0$ due to $\langle A_t\rangle = \langle A\rangle = 0$. We work with random initial states, hence in the linear response regime (sufficiently small δ) we have an identity (29) connecting the state-averaged purity fidelity decay to the purity correlator. Asymptotic ($\delta \to 0$) exactness of this relation has been carefully checked in all different cases of interest since it provided a crucial test of our numerical procedures.

In the first set of numerical calculations, we check the decay of correlations, either with respect to the full trace (non-reduced) inner product or with respect to the reduced inner product, while in the second set of calculations we explicitly compute the purity fidelity decay $\langle F_{\mathrm{P}}(t)\rangle_{\psi}$ and compare the exponents with our predictions based on correlation decay. Calculations are performed with three different system sizes, namely $L = 12, 16, 20$. Since we want the perturbation strength to have certain size L-independent effects we scale it by fixing $\delta' = \delta\sqrt{L/L_0}$ when varying L where we choose $L_0 := 24$. For example, in such a case the predicted exponents of the fidelity decay [9] do not depend on L.

In figure 1 we examine the decay of non-reduced correlations of the magnetization $C(t) = \langle MM_t\rangle/L$ for the three cases: (*a*) integrable $h_z = 0.0$, (*b*) intermediate $h_z = 0.4$ and (*c*) ergodic and mixing $h_z = 1.4$. We find that $C(t)$ has a non-vanishing plateau (stiffness $\bar{C} = D/L$) in the integrable and intermediate non-integrable case, where the correlation

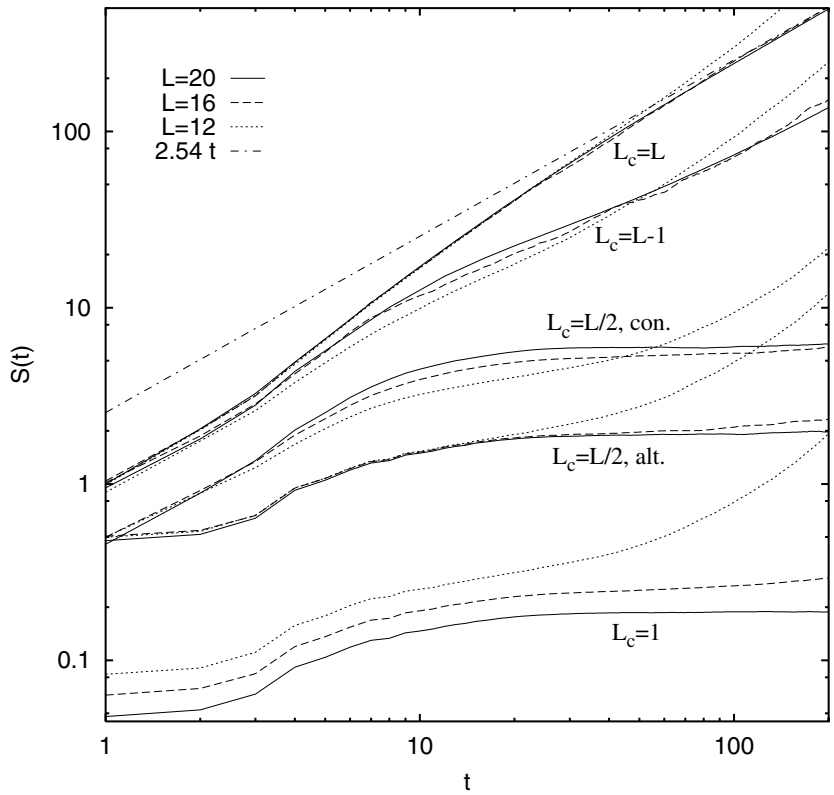

Figure 2. Partial correlation sums in the mixing regime $h_z = 1.4$ for different divisions $\mathbb{Z}_L = \mathcal{J} \cup \mathcal{J}'$. The structure (alternating A versus continuous C) and/or the number L_c of elements of the set $\mathcal{J}'$ is indicated in the label near each triple of curves (full for $L = 20$, dashed for $L = 16$ and dotted for $L = 12$).

function in the integrable case agrees excellently with the analytical result for $L \to \infty$ [21], while in the ergodic and mixing case we find exponential decay of correlations. We note an interesting distinction between correlation decays in integrable and intermediate cases, namely in the integrable case (a) the relaxation of $C(t)$ towards the plateau value D/L is a power law $\sim t^{-3/2}$ whereas in the intermediate case (b) it looks like an exponential (see inset of figure 1(b)).

In figure 2 we show partial correlation sums $S_{\mathcal{J}}(t)$ for the ergodic and mixing case (c) for different reducing sets $\mathcal{J}$ appearing in the three cases (A, C, S) of divisions. We note that $S_{\mathcal{J}}(t)$ increases linearly in t only for the non-reduced case $\mathcal{J} = \emptyset$ where the (non-reduced) correlation function is homogeneous in time, whereas in all other cases $S_{\mathcal{J}}(t)$ increases slower than linear, in fact in most cases it quickly saturates to a maximum value which does not increase with increasing size L. Therefore, for large system sizes L in the ergodic and mixing situation the purity correlator becomes determined by the total (non-reduced) correlator only (confirming equation (27)), namely $S_{\mathrm{P}}(t) = st$, where $s = 2.54$. This is illustrated in figure 3.

In figures 4 and 5 we show analogous results on partial correlation sums and purity correlators in the integrable case (a). Here, $S_{\mathrm{P}}(t) \propto t^2$. Numerical results are compared with analytical expressions (55) and (56), and the agreement is very good. In figure 6 we also

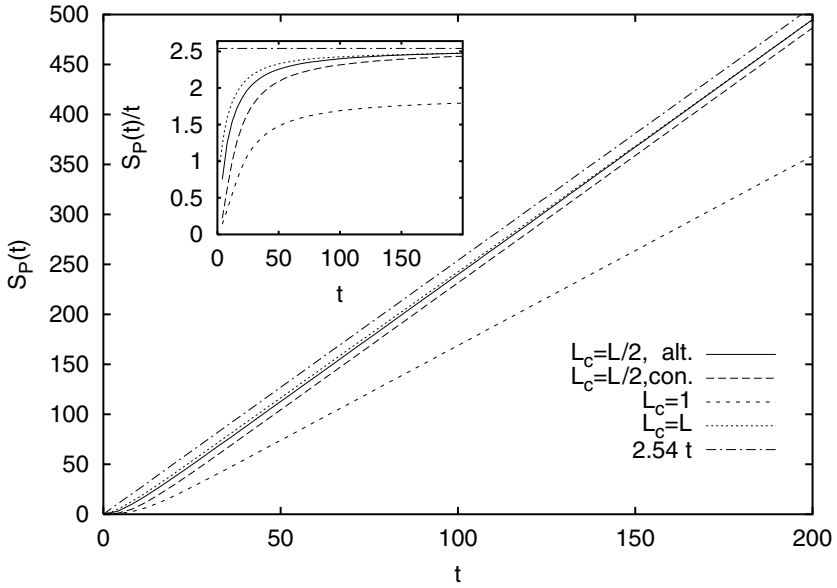

Figure 3. Purity correlator $S_{\mathrm{P}}(t)$ in the mixing regime $h_z = 1.4$ for different cases (A, C, S and total (non-reduced) correlator, for full, long-dashed, short-dashed and dotted curves, respectively) as compared to the asymptotic linear increase (chain line).

show the purity correlators in the intermediate regime (b) of non-ergodic and non-integrable dynamics, where the results are, as expected, qualitatively very similar to the integrable case (a), namely we have the quadratic growth of purity correlators $S_{\mathrm{P}}(t) \propto t^2$.

In the second set of numerical experiments, we calculate the purity fidelity $F_{\mathrm{P}}(t)$ with respect to a random initial state $|\psi\rangle$ and average the result over a sufficient number of realizations of random initial states such that statistical fluctuations are negligible. We note that $F_{\mathrm{P}}(t)$ is *self-averaging*, namely by increasing the dimension of the Hilbert space, i.e. increasing the size L, the purity fidelity of a single random initial state converges to the state average $\langle F_P(t)\rangle_\psi$. Here we are interested not only in the linear response regime but also in the global behaviour of purity fidelity, so we chose to display the results at the (scaled) perturbation strength $\delta' = 0.01$ for sufficiently long times such that the purity fidelity drops several orders of magnitude and approaches the plateau F^* (34).

According to our relation (29) and the behaviour of purity correlators, we predict (and find!) that purity fidelity decay is faster in integrable and intermediate cases (a) and (b) than in the ergodic and mixing case (c). This result is most clear-cut in the case of division (A). One should see figure 7 for a comparison of purity fidelity decay in the integrable and mixing cases. We have found very good agreement with the predicted exponential decay (26) in the mixing case (c), whereas in the integrable regime (a) numerical results suggest a global Gaussian decay of purity fidelity (similar to the Gaussian fidelity decay found in [9]) with an analytically computed exponent (57). Due to the finite Hilbert space dimension, we find a saturation of purity fidelity for very long times at the plateau value which has been computed in equation (34). In order to avoid the trivial effect of a constant term F^* in (33) we subtract it and in the following plots show the quantity $|F_{\mathrm{P}}(t) - F^*|/(1 - F^*)$. In figure 8 we show that purity fidelity decay in the ergodic and mixing regime (c) is independent of the type of division (A, C, S). In addition, we show in the same figure the scaling of purity fidelity in the

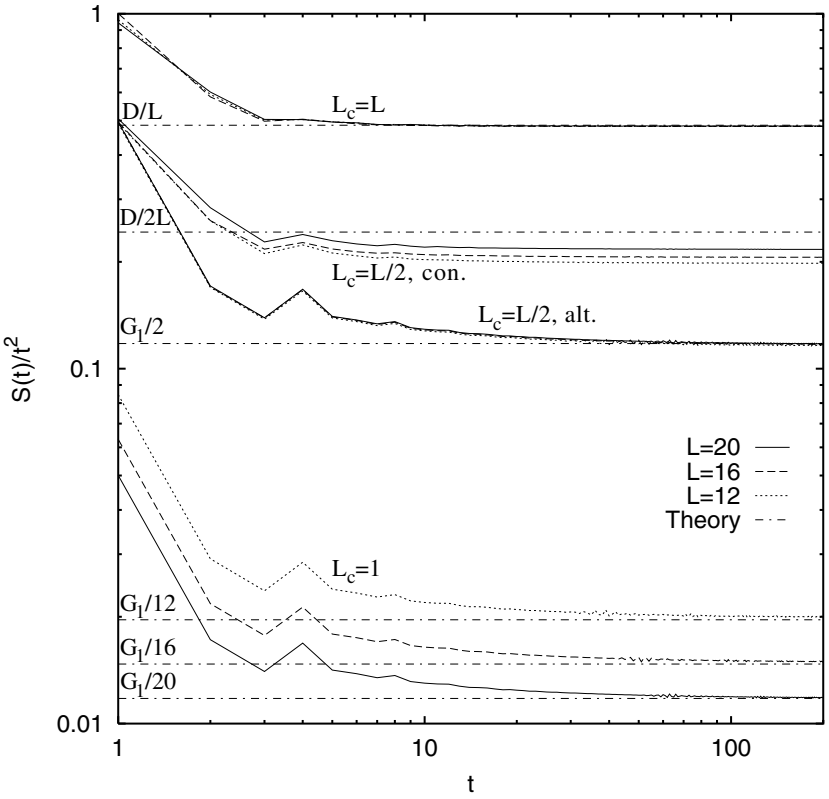

Figure 4. Partial correlation sums in the integrable regime $h_z = 0.0$ compared to the analytically computed coefficients for different divisions indicated by labels describing the structure (continuous/alternating) and number of elements of the set $\mathcal{J}'$.

thermodynamic limit: for fixed value of the scaled perturbation parameter δ' the results do not depend on the size L (for large L) provided we scale proportionally the sizes of division sets $\mathcal{J}_c$ and $\mathcal{J}_e$. In figures 9 and 10 we show an analogous comparison of purity fidelity decay in integrable and intermediate cases (a) and (b) for different divisions and show that, in all cases, short time behaviour is well reproduced by the linear response coefficients given by purity correlators. For longer times a global Gaussian behaviour with theoretical exponents works quite well (in particular for the case (A) where again a nice scaling with the size L approaching the thermodynamic limit was observed). Results for purity fidelity in the intermediate case (b) (figure 10) are qualitatively the same as in the integrable case (a) (figure 9). This is expected based on the same behaviour ($\propto t^2$) of correlation sums (compare figures 5 and 6).

4. Discussion

We have analysed the properties of decoherence in the framework of the question of reversibility with a perturbed time reversed Hamiltonian. Up to now only the state correlation function, usually called fidelity, has been discussed in this context. The calculations were carried out in the framework of partial tracing over the environment and unitary time evolution.

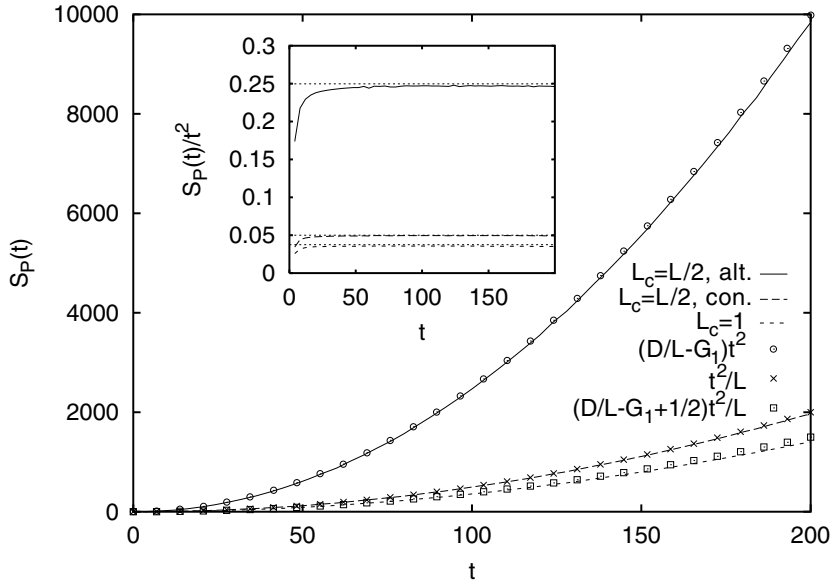

Figure 5. Purity correlator $S_P(t)$ in the integrable regime $h_z = 0.0$ compared with analytical expressions (sampled symbols) for different cases of divisions (A, C, S). In the inset we emphasize the quadratic increase by plotting $S_P(t)/t^2$ and comparing to theoretical coefficients (56) (dotted).

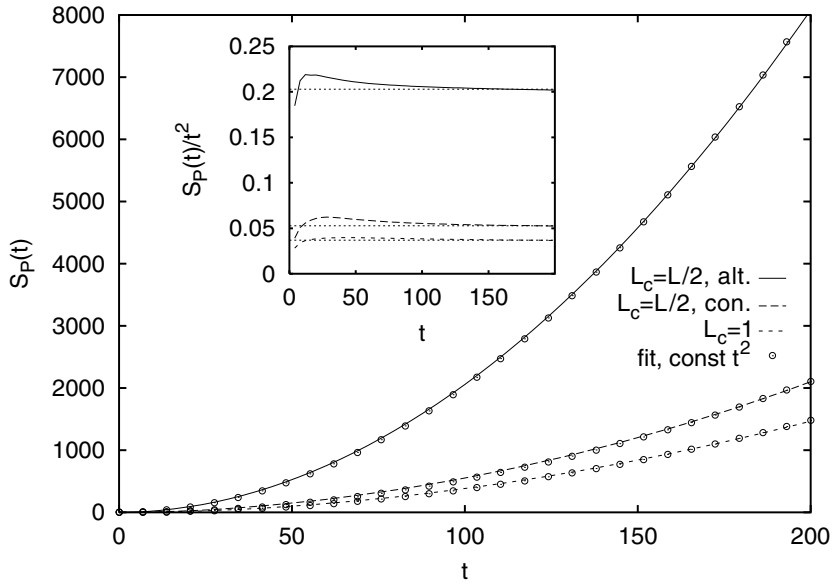

Figure 6. Same as in figure (5) but for the intermediate case (*b*). Since we have no analytical predictions here, we compare the data by best fitting quadratic functions (sampled symbols—circles and dotted lines in the inset) whose coefficients are later used for comparison to purity fidelity decay.

The remaining density operator on the subspace was analysed in terms of the trace of its square, usually called purity. It is more convenient to use this quantity rather than entropy

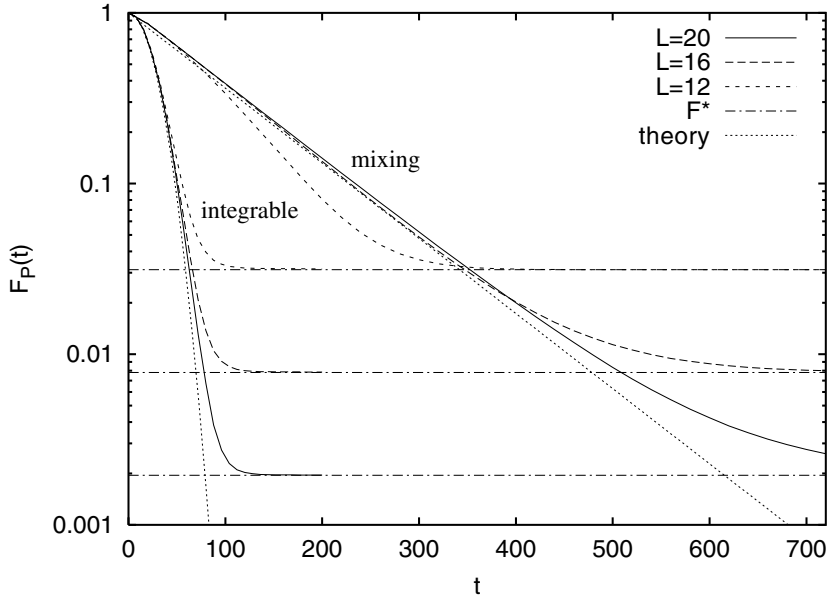

Figure 7. Purity fidelity decay: comparison between mixing $h_z = 1.4$ and integrable regime $h_z = 0$, at scaled perturbation $\delta' = 0.01$. Dotted curves give the predicted exponential (26) in the mixing regime and a suggested Gaussian with a theoretically computed timescale (57) in the integrable regime, while horizontal chain lines give the plateau values (34).

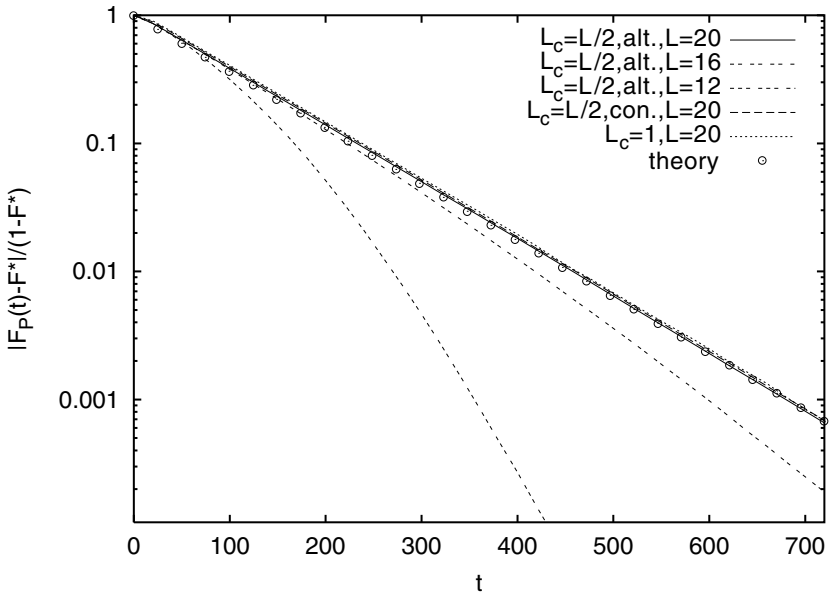

Figure 8. Purity fidelity decay in the mixing regime $h_z = 1.4$ for different types of division and different sizes L (indicated in the legend) and $\delta' = 0.01$. Theoretical decay (for $L \to \infty$) is given by sampled symbols (circles).

itself, because its analytic form allows explicit calculations to be performed to a much larger extent than would be the case for the entropy, where the logarithm complicates things.

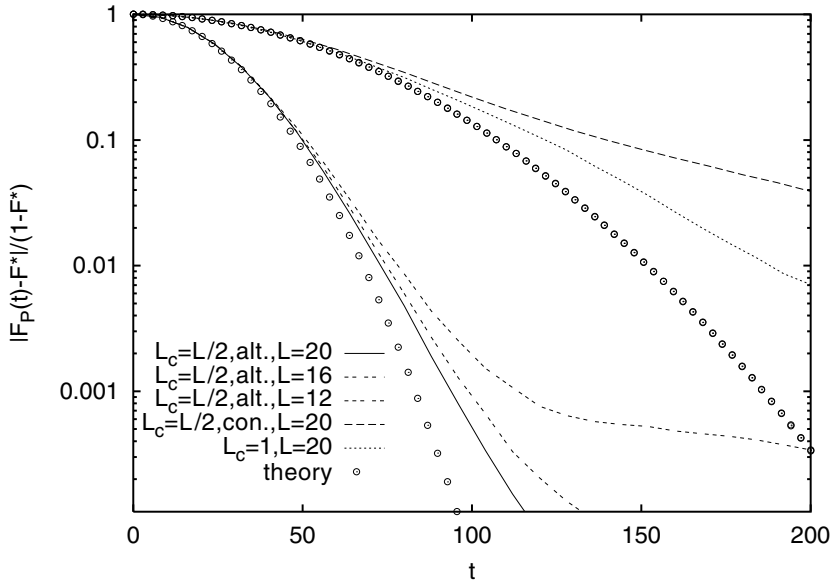

Figure 9. Same as in figure 8 but for the integrable regime (*a*). Note that the theoretical (Gaussian extrapolated) curves for cases (C, S) are practically indistinguishable.

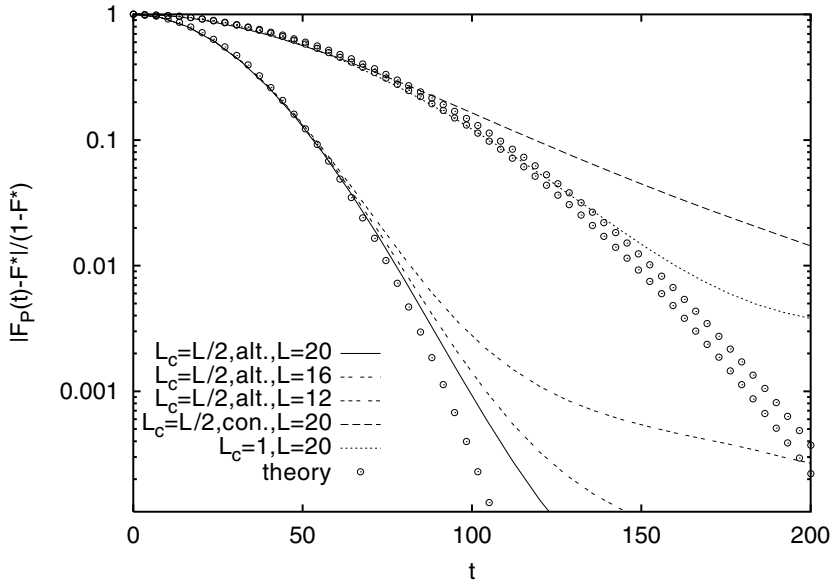

Figure 10. Same as in figure 8 but for the intermediate regime (*b*).

We apply the techniques developed for the calculation of fidelity to a Hilbert space which is a product space. Keeping track of the respective indices we can readily perform the partial traces needed, and find that they relate to slightly modified correlation functions of quantum observables. This allows the results for purity fidelity to be compared with those of fidelity [9].

As for the fidelity, it becomes apparent that the decay time of correlations is a relevant short timescale, and this induces for mixing systems an exponential decay with a timescale that is mainly related to the strength of the perturbation. This implies a linear decay after the end of the trivial short time 'perturbative' regime (related to the quantum Zeno effect) which always leads to a quadratic decay. For integrable systems, on the other hand, correlations do not decay and thus the quadratic decay of purity fidelity survives beyond the perturbation regime. This amounts to the central result that decoherence will be *faster* for an integrable system than for a chaotic one in strict analogy to the findings for fidelity.

The techniques developed are applied to kicked spin chains where analytic results for the integrable case and numerical results for the correlation functions of observables can easily be obtained. In the thermodynamic limit, we obtain exponential decay of purity fidelity for the mixing case and Gaussian decay for the integrable case with uniform coupling to the environment, which is simulated by assigning alternatively one spin to the environment and one to the central system. If the spin chain is simply cut in two or if a single spin is associated with the central system we find deviations from the Gaussian shape, but the decay rate associated with the quadratic behaviour at short but non-perturbational times is correctly reproduced thus maintaining the fact that even in these situations coherence decays faster in the integrable case than in the mixing one.

The central message is thus that decoherence in echo situations follows fidelity. Indeed, the reversibility of decoherence under perturbation of the reversed Hamiltonian is better for mixing systems than for integrable ones, and the timescales on which this occurs depend sensitively on the strength of this perturbation.

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