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Spectral statistics of a system with sharply divided phase space

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Abstract

We construct a family of non-Kolmogorov–Arnold–Moser (non-KAM) piecewise-linear continuous 2D area-preserving maps which have sharply divided phase space with regions of regular elliptic and chaotic hyperbolic motion. For such systems the shape of the islands of regular motion is either a solid ellipse or a solid polygon, depending on the (ir)rationality of the frequency, and thus the total area of the regular region of phase space can be computed analytically or at least rigorously estimated from below. We analyse the spectral statistics for a few examples of the quantization of our maps, and show that they provide a convenient 'playground' for testing and confirming the key assumptions required for the validity of Berry–Robnik formulae.

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1. Introduction

The energy level statistics of quantum systems whose classical counterparts have mixed phase space made up of coexisting regular and chaotic regions is still a matter of lively debate (see e.g. Ketzmerick *et al* (2000), Makino *et al* (1999, 2000), and Prosen (1998)). The asymptotic semiclassical formulae were derived a while ago by Berry and Robnik (1984) for the nearest-neighbour level spacing distribution and by Seligman and Verbaarschot (1985) for the long-range two-point spectral statistics. However, from rigorously, even in classical mechanics the coexistence of regular and chaotic motion is an open problem (Strelcyn 1991).

The main problem is that the two key assumptions required for the validity of the Berry–Robnik-type formulae, namely: (I) the statistical independence of the regular and chaotic spectral subsequences; and (II) absence of dynamical localization and other mechanisms for deviation from random-matrix spectral statistics of 'chaotic levels' (see e.g. Bohigas *et al* (1993)), become valid only in the very 'far' semiclassical regime of very small effective values of Planck's constant. There are two essential obstacles to the validity of the key assumptions ((I), (II)) in generic Hamiltonian dynamical systems at numerically or experimentally accessible values of $\hbar$: (A) the complex structure of the infinite number of regular islands which form a fractal boundary with the chaotic sea; and (B) the abundance of cantori and other 'sticky' invariant objects in phase space which form partial barriers to transport on chaotic components. It is worth mentioning that even the numerical calculation of just the fractional volume of the regular region of classical phase space is in general a difficult problem (Robnik *et al* 1997, Prosen and Robnik 1998). The first statistically significant demonstrations of the Berry–Robnik formulae thus came rather late (Prosen and Robnik 1994, Prosen 1995, 1998), since one typically needed extremely large sequential quantum numbers in order to be in the so-called far semiclassical regime. Therefore, at practically accessible energies one will generically find, apart from regular and chaotic eigenstates 'living' on regular and chaotic phase-space components, respectively, also the so-called *hierarchical* states which are localized by the cantori near the boundaries between the regular islands and the chaotic components (Ketzmerick *et al* 2000). However, it is clear that the relative fraction of such hierarchical states will decrease when approaching the semiclassical limit, so in the ultimate semiclassical regime—though perhaps practically irrelevant—the simple Berry–Robnik theory will apply.

The central motivation for the present paper is to provide an example of a (toy) dynamical system where assumptions (I), (II) are easy to control and verify, thus making the Berry–Robnik theory self-evident, even in the so-called 'near' semiclassical regime. On the other hand, we want to explore the possibility of reducing the complexity of classical phase space, by violating the assumptions of the Kolmogorov–Arnold–Moser (KAM) theorem (Moser 1973) in a minimal way, namely by taking a potential which is a C^1- but not a C^2-function of coordinates. In order to do this we study a kicked-rotor-type map on the unit torus $[0, 1) \times [0, 1)$, where the kicking function is continuous but not continuously differentiable:

$$\begin{aligned} y_n &= y_{n-1} + \varepsilon(1 - |2x_{n-1} - 1|) \qquad &(\mathrm{mod}\ 1) \\ x_n &= x_{n-1} + y_n \qquad &(\mathrm{mod}\ 1). \end{aligned} \tag{1}$$

x and y are the angle and momentum variables, respectively, and we can assume without loss of generality that $\varepsilon > 0$. The map (1) will be called a continuous sawtooth map (CSM), and its properties are fundamentally different from the *hyperbolic* discontinuous sawtooth map which has a long history (Dana *et al* (1989); see also, Devaney (1989), Troll (1991, 1992)). However, we should mention that also a mixed, *elliptic–hyperbolic* continuous piecewise-linear map, namely the 'linearized standard map', which is very similar (but not identical) to (1), has been studied by Scharf and Sundaram (1991, 1992). In relation to this, one should note an additional constant in the potential of (1) which changes the character of the primary fixed points (and also the global appearance of phase-space 'portraits') with respect to those of the works of Scharf and Sundaram (1991, 1992) where the classical and quantum aspects of homoclinic tangles around the hyperbolic fixed point were studied.

It is convenient to decompose the map (1) as $M = M_{free} \circ M_{kick}$ where $M_{free}(x, y) = (x + y, x)$, $M_{kick}(x, y) = (x, y + \varepsilon(1 - |2x - 1|))$, and to write its symmetric version as $M' = M_{kick}^{(1/2)} \circ M_{free} \circ M_{kick}^{(1/2)}$.

2. Properties of the classical continuous sawtooth map

CSM (1) has a very unusual non-KAM-like phase-space structure that immediately caught our attention. Though it is locally always linear, it is neither purely elliptic nor purely hyperbolic if $\varepsilon < 2$, since the trace of the Jacobian flips between $2 + 2\varepsilon$ and $2 - 2\varepsilon$ for $x < 1/2$ and $x > 1/2$, respectively. The primary fixed point $(0, 0)$ is neither elliptic nor hyperbolic since it lies on the boundary between the two regions—the cut—which consists of two lines, namely $x = 0$, or equivalently $x = 1$, and $x = 1/2$. It is easy to convince oneself that the existence of

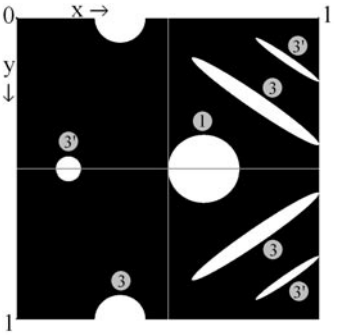


Figure 1. The chaotic component of a symmetric CSM′ for $\varepsilon = 1.31$. White regions are the regular elliptic islands belonging to one stable 1-cycle and two stable 3-cycles (islands belonging to different cycles are denoted by different labels (1, 3, or 3′)).

regular motion in the CSM will depend on the existence of stable cycles of the map, while for $\varepsilon > 2$ the map is *uniformly hyperbolic*, since then all periodic orbits (cycles) are hyperbolic.

The cycles of the CSM, whose points denoted by z^* satisfy $M^k z^* = z^*$, can be computed systematically for small integer values of k by solving a certain system of linear equations, with inequality constraints due to the piecewise linearity of the map. The stability of a k-cycle $\alpha = \{M^j z^*, j = 0, 1, k-1\}$ is determined from its monodromy matrix

$$\boldsymbol{T}_\alpha = \boldsymbol{T}_{b(M^{(k-1)}z^*)} \cdots \boldsymbol{T}_{b(Mz^*)} \boldsymbol{T}_{b(z^*)} \tag{2}$$

which depends only on a sequence of binary symbols, $b(z^*) = L$ or R, if $x(M^j z^*) < 1/2$ or $x(M^j z^*) > 1/2$, respectively, and

$$\boldsymbol{T}_{\substack{L\\R}} = \begin{bmatrix} 1 \pm 2\varepsilon & 1 \\ \pm 2\varepsilon & 1 \end{bmatrix}. \tag{3}$$

The cycle α is stable if $|\mathrm{Tr}\, \boldsymbol{T}_\alpha| < 2$. Then the points $M^j z^*$ sit at the centres of k elliptic islands where the motion is entirely determined by a linear map $\boldsymbol{T}_\alpha$. The sizes of the elliptic islands i.e. the *boundary ellipse*, can be determined from the simple condition that at least one of the islands should touch the cut $x = 0$, 1/2, or 1, while beyond the cut the symbolic code determining the linear map changes and the motion typically becomes chaotic (see figure 1 for an illustration). One can easily write an explicit equation for the family of k ellipses around any point z^* of a stable k-cycle α and also for the occupied phase-space area ρ_α. Now the question arises of what happens if we launch a trajectory just outside but near the boundary ellipse of an island around cycle α? Does it remain trapped in the vicinity of the elliptic island or shoot off into the chaotic sea?

It turns out that the answer to this question depends sensitively on the frequency ω, where $\exp(\mathrm{i}\omega)$ is an eigenvalue of T_α. If frequency ω is irrational, then there is typically a narrow strip around the elliptic island of KAM-like structures with many stable satellite islands (see figure 2 for an example with $\varepsilon = 1.3$). The width of this elliptic strip is typically small and can vary considerably; for example, it can be practically vanishing like in the case of $\varepsilon = 1.31$ shown in figure 1. On the other hand, if the frequency is rational, $\omega = 2\pi m/n$, then the motion around the k-cycle is periodic with the period $p = nk$. In this case, the motion on the outer ellipses beyond the boundary ellipse can also be stable, provided that all the points of such a p-cycle lie on the same sides of the cut as the corresponding points of the original k-cycle. The set of all such points consists of k n-sided (squeezed regular) polygons drawn around the boundary ellipses each having one of its sides lying along one of the cuts $x = 0, 1/2, 1$. However, all the orbits

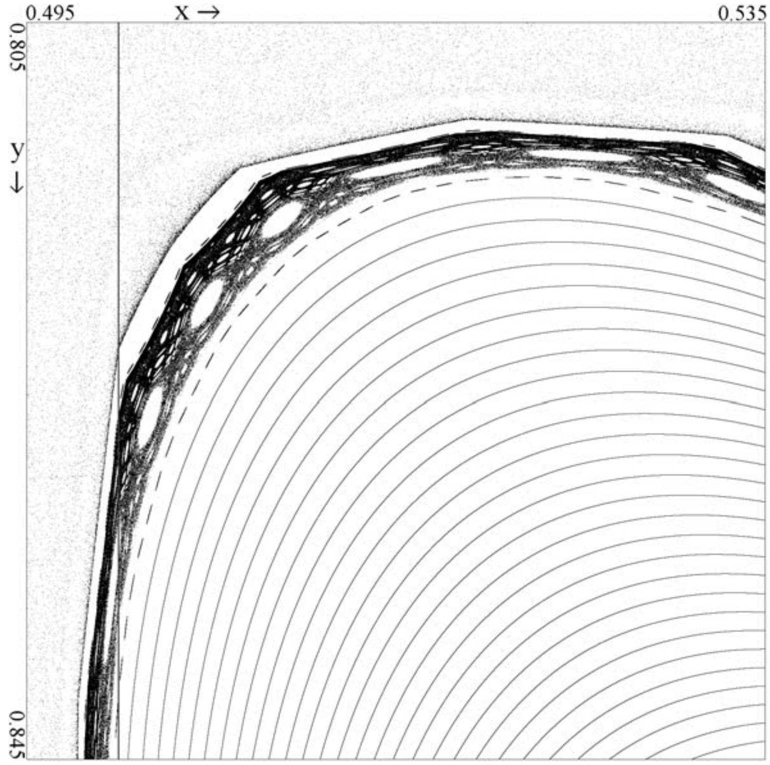


Figure 2. A magnified view of the region near a 1-cycle elliptic island with $\varepsilon = 1.3$ (slightly different to that in figure 1). The straight vertical line is the cut at $x = 1/2$. We show parts of 40 different trajectories consisting of 3200 000 iterations each.

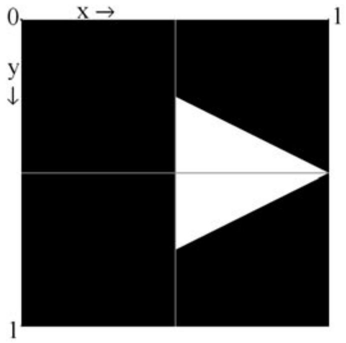


Figure 3. The single solid chaotic component for the case with $\varepsilon = 3/2$.

launched outside the stable polygons turn out to be chaotic, and quickly fill the chaotic region. Therefore, in the resonant case of rational frequency ω, the regular islands have the shapes of polygons with a clear-cut boundary with the chaotic region. In figure 3 we show an amusing example for $\varepsilon = 3/2$ with one stable 1-cycle with $\omega = 2\pi/3$ yielding a single regular triangular region of area $\rho = 1/8$, whereas its complement is a single solid chaotic component. Non-existence of possible higher stable cycles or islands has been carefully numerically checked.

Let us conclude this section by making some global numerical observations. We were almost always able to find at least one stable cycle in the range $0 < \varepsilon < 2$; however, the length of the shortest existing stable cycle grows as ε decreases. Since the sizes of the corresponding islands decrease with increasing cycle length, one may predict an approach to a kind of slowly ergodic, diffusive regime as $\varepsilon \to 0$. Another notable observation is that the total number of stable cycles having islands with significant area typically turns out to be quite small and finite; therefore the global phase-space structure appears much 'cleaner' than in the typical KAM situation.

3. Quantum spectral statistics and the Berry–Robnik formula

We quantize the symmetrized CSM map M' on a finite Hilbert space of dimension N using the standard procedure (see e.g. Prosen and Robnik (1994) where the same notation is used):

$$U_{kk'} = \langle x_k|\hat{U}|x_{k'}\rangle = \exp\bigl(2\pi \mathrm{i} N\bigl(\tfrac{1}{2}(x_k - x_{k'})^2 + \tfrac{1}{2}\varepsilon h(x_k) + \tfrac{1}{2}\varepsilon h(x_{k'})\bigr)\bigr) \tag{4}$$

$$h(x) = \begin{cases} x^2 + \frac{1}{4}, & x < \frac{1}{2} \\ -x^2 + 2x - \frac{1}{4}, & x > \frac{1}{2}. \end{cases} \tag{5}$$

Here, $|x_k\rangle$, with $x_k = (k-1/2)/N$, $k = 0, 1, \ldots, N-1$, is a complete basis of discrete position states. We compute the quasienergy spectrum $\{\varphi_n, n = 1, \ldots, N\}$ by simply diagonalizing the unitary matrix $U|n\rangle = \exp(\mathrm{i}\varphi_n)|n\rangle$. Assuming that the sequence of quasienergies is ordered, we define a sequence of normalized level spacings as $S_n = N(\varphi_{n+1} - \varphi_n)/2\pi$. Here we are considering the integrated level spacing distribution $W(S) = \#\{S_n < S; n = 1, 2, \ldots, N\}/N$, which is the probability that the randomly chosen spacing is less than S.

Berry and Robnik (1984) have proposed a formula for the level spacing distribution of systems with mixed phase space. On the basis of assumptions (I) and (II), Berry and Robnik have decomposed the spectrum into statistically independent Poissonian and GOE/GUE spectral samples, corresponding to the regular and chaotic phase-space regions, respectively. Since the gap probability $E(S)$ that there is no level in a random spectral interval of size S factorizes upon statistically independent superposition of the level sequence, one can write

$$E_{BR}(S) = E_{Poisson}(\rho_1 S) E_{GOE}(\rho_2 S) \tag{6}$$

for the simplest two-component case, where ρ_1 is the relative phase-space volume of all regular elliptic islands and $\rho_1 + \rho_2 = 1$. The integrated level spacing distribution $W_{BR}(S)$ is related to the gap distribution $E_{BR}(S)$ through the equation

$$W_{BR}(S) = 1 - \frac{\mathrm{d}E_{BR}(S)}{\mathrm{d}S}. \tag{7}$$

We calculated the level spacing distribution of our mapping for various values of the parameter ε and compared the results with the values obtained with the Berry–Robnik formula. The value of the parameter ρ_1 was calculated analytically from all of the areas of regular elliptic islands in the classical phase space. Of course, we obtained the best results for the cases where the area of the boundary KAM-like regions (rings) around the elliptic islands that we neglected was the smallest. We note that regular levels belonging to islands around any stable cycle can be calculated in the semiclassical approximation by quantizing a harmonic oscillator sited at the centre of the island chain, giving $\varphi_n^{reg} = \varphi_0^{reg} + \omega n$. This clearly generates a Poissonian subspectrum for irrational ω and a highly degenerate subspectrum for rational ω. In figure 4 we present the results for the case in figure 1 with $\varepsilon = 1.31$ where $\rho_1 = 0.122\,783$, which agrees excellently with the Berry–Robnik formula.

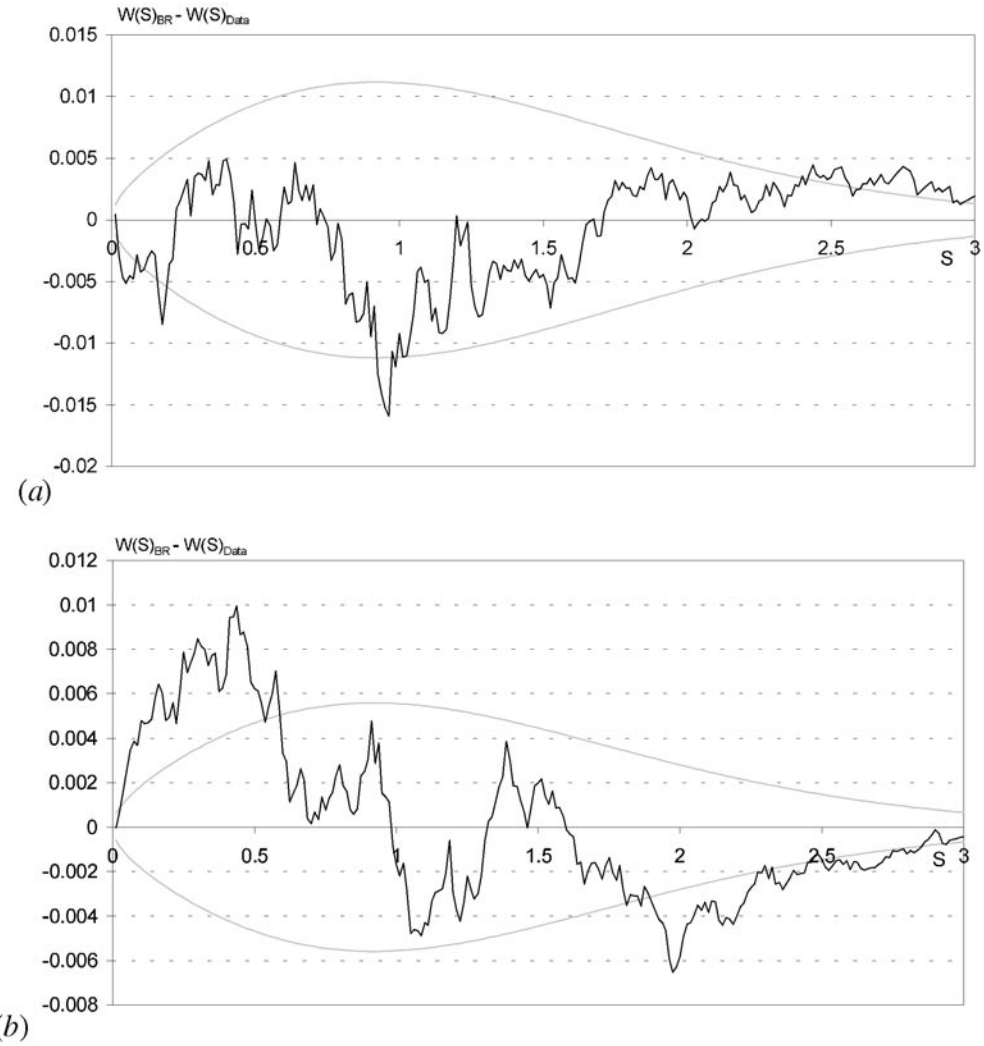
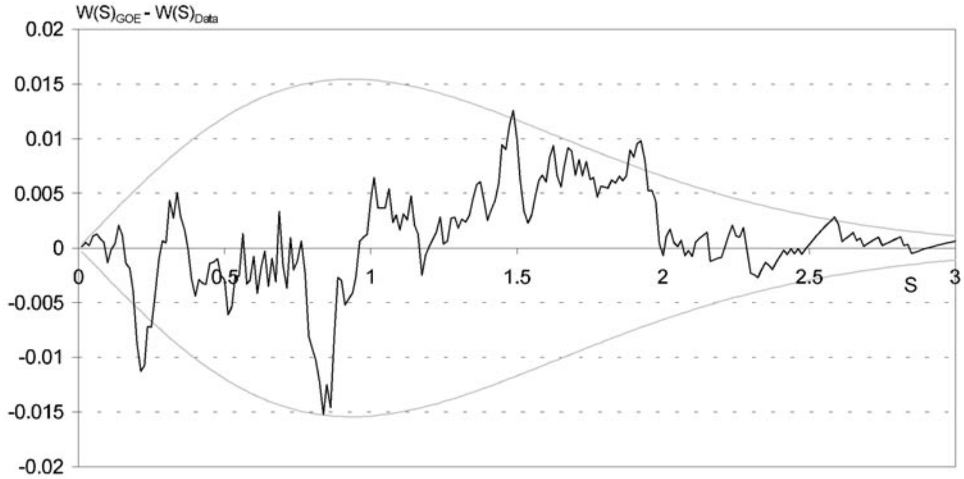


Figure 4. We show the deviation of the integrated level spacing distribution from the Berry–Robnik formula for the case of $\varepsilon = 1.31$ and $N = 2000$ (*a*), $N = 8000$ (*b*). Note that deviation is very small in absolute units and around the expected one-sigma error band ($\pm\sqrt{W(S)(1 - W(S))/N}$).

Perhaps even more interesting is the quantization of the case with a single sharp triangular island with $\varepsilon = 3/2$. Here we use the (semiclassically approximate) degeneracy of the regular levels in order to separate the regular and the chaotic (sub-) spectrum. Indeed, the total fraction of regular (degenerate) levels turns out to be very close to the semiclassical value $\rho_1 = 1/8$ which is the classical area of the triangular island. Furthermore, the statistics of chaotic levels turns out to be in statistical agreement with the infinite-dimensional GOE prediction, as shown in figure 5.

4. Conclusions

We have presented an interesting toy model of dynamical systems, a family of kicked-rotor-type piecewise-linear area-preserving maps, which violate the conditions of the KAM theorem



(a)

(b)

Figure 5. We show the deviation of the integrated level spacing distribution of the chaotic subsequence for the triangular case with $\varepsilon = 1.5$ and $N = 1200$ (*a*), $N = 8000$ (*b*) from the expected GOE distribution. Again, the deviation is very small and within the expected statistical error.

in a minimal way. We constructed examples with mixed phase space with a simple and sharp boundary between regions of regular and chaotic motion where we can calculate their relative measures analytically. We applied our map to study quantum level statistics, where we showed most clearly how the key assumptions required in the derivation of the Berry–Robnik formula are clearly satisfied due to the simple phase-space structure.

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