Conservation laws in the one-dimensional Hubbard model

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We examine the nature, number, and interrelation of conservation laws in the one-dimensional Hubbard model. In previous work by Shastry [Phys. Rev. Lett. **56**, 1529 (1986); **56**, 2334 (1986); **56**, 2453 (1986); J. Stat. Phys. **50**, 57 (1988)], who studied the model on a large but finite number of lattice sites (N_a), only N_a+1 conservation laws, corresponding to N_a+1 operators that commute with themselves and the Hamiltonian, were explicitly identified, rather than the $\sim 2N_a$ conservation laws expected from the solvability and integrability of the model. Using a pseudoparticle approach related to the thermodynamic Bethe ansatz, we discover an additional N_a+1 independent conservation laws corresponding to *nonlocal*, mututally commuting operators, which we call *transfer-matrix currents*. Further, for the model defined in the whole Hilbert space, we find there are two other independent commuting operators (the squares of the η-spin and spin operators) so that the total number of local plus nonlocal commuting conservation laws for the one-dimensional Hubbard model is $2N_a+4$. Finally, we introduce an alternative set of $2N_a+4$ conservation laws which assume particularly simple forms in terms of the *pseudoparticle* and *Yang-particle* operators. This set of mutually commuting operators lends itself more readily to calculations of physically relevant correlation functions at finite energy or frequency than the previous set.

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I. INTRODUCTION

The one-dimensional Hubbard model, arguably the simplest physical model for N interacting electrons on a lattice of N_a sites, is well known to be solvable by the coordinate Bethe ansatz.[1–3] Lieb and Wu reduced the problem of diagonalizing the model to solving a set of two coupled nonlinear equations.[1] Takahashi[2] introduced the *string hypothesis*, which led to the thermodynamic Bethe-ansatz-coupled nonlinear equations (also see Ref. 2,3). While the two Lieb-Wu equations are associated with two types of *rapidities*, each thermodynamic Bethe ansatz equation of Takahashi is also associated with a different type of rapidity, whose number becomes infinite in the limit of $N_a\rightarrow\infty$. The string hypothesis is only valid in the limit of very large N_a which is associated with the thermodynamic Bethe-ansatz equations.

Reference 4 considered Takahashi's thermodynamic Bethe-ansatz equations, and associated each of the corresponding rapidities with a branch of *pseudoparticles*. The concept of a pseudoparticle is well defined only within the thermodynamic Bethe ansatz equations and follows from the ideal positions of the discrete *pseudomomentum* quantum numbers associated with the string-hypothesis rapidities. The pseudoparticles are neither fermions nor bosons, but they do obey an anticommuting algebra. As a result of the integrability of the Hubbard model, their collisions are dissipationless and do not lead to momentum and/or energy transfer, with pseudoparticle collisions producing only shifts in their phases.[4,5] The pseudoparticle operator representation of the thermodynamic Bethe-ansatz solution provides insight into the low- and finite-energy elementary excitations of the model and associated symmetries[6] and is useful for the study of correlation functions at finite energy and frequency.[5,7]

Shastry[8] explored the relationship between the solvability of the model and the concept of *integrability* (inspired by Liouvilian integrability in classical mechanics) by exhibiting an infinite number (in the thermodynamic limit, $N_a\rightarrow\infty$) of commuting operators corresponding to the expected conservation laws. He also found a two-dimensional classical vertex model whose transfer matrix commutes with the Hubbard Hamiltonian. These results were verified by other approaches.[9–12] The invariance of the model under two Y[su(2)] Yangians was also studied.[13,14]

As an alternative to the coordinate Bethe ansatz, the solution of many integrable quantum models can also be obtained by means of the inverse scattering method, first introduced in Ref. 15. This is the quantum version of the inverse scattering method used in classical problems. This method provides more insight into the structure and symmetries of integrable systems than the usual Bethe-ansatz solution. However, as a consequence of the nonadditive property of the Hubbard-model R matrix, its solution by means of the inverse scattering method has remained a long-standing open problem.

Recently, Martins and Ramos solved this problem by first finding the commutation rules of the operators present in the embedding vertex model found by Shastry, and then solving the one-dimensional Hubbard model by means of the inverse scattering method.[16,17] They found that a hidden symmetry of six-vertex type was important for the integrability of the model.

Within the inverse-scattering solution of the model, the N_a-1 local conservation laws first identified by Shastry can be extracted from the transfer matrix. This follows from the fact that the associated auxiliary eigenvalue is analytic in the spectral parameter λ. The N_a-1 coefficients of the λ expan-

sion of the logarithm of that eigenvalue provide the expressions in terms of Bethe-ansatz rapidities for the corresponding N_a-1 Shatry's conservation laws.[17] Hence, in this paper we call the N_a-1 conservation laws found in Refs. 8–11 and 17, *transfer-matrix charges*.

However, there remain important open issues regarding the nature, number, and interrelations of the conservation laws in the one-dimensional Hubbard model. For instance, what is the total number of independent conservation laws? For a large but finite number of lattice sites N_a, the total number of transfer-matrix charges is N_a+1, when one includes the obvious conservation of total electron number and magnetization. This number appears insufficient to ensure the integrability of the model, which by analogy to classical mechanics ought to require a *minimum* number of commuting conservation laws equal to the number of degrees of freedom, which is $2N_a$, since there are two degrees of freedom, spin- and spin-down electrons, per lattice site.

In this paper we provide insights into that open issue. We first show that combining the expressions of the transfer-matrix charges in terms of the Bethe-ansatz rapidities provided by the transfer-matrix approach[17] on the one hand, with the thermodynamic Bethe ansatz string hypothesis of Takahashi[2,3] on the other hand, allows us to express these conservation laws in the pseudoparticle basis. Then, observing that the numbers of right and left pseudoparticles are *independently conserved*, we find explicit expressions for an extra set of N_a+1 new conservation laws beyond the transfer-matrix charges previously found by Shastry; we term the operators corresponding to these new conservation laws *transfer-matrix currents*. These conservation laws are *nonlocal* when expressed in terms of electron operators, with their nonlocality providing a natural explanation for their not having been previously found (to our knowledge).

While we know how to express the transfer-matrix *charges* both in terms of pseudoparticle and electron operators, we have the expression of the newly found transfer-matrix *currents* only in terms of pseudoparticle operators. For general values of the onsite repulsion U, we have been unable to find explicit expressions for them in terms of electron operators. Hence, to gain insight, we study the limit $U/t\rightarrow 0$, where as usual t is the transfer integral. In this limit, we obtain explicit expressions for the conservation laws corresponding to the first few transfer-matrix currents, demonstrating that they are indeed nonlocal in terms of electron operators in this limit.

The Bethe ansatz solution refers to the Hilbert subspace spanned the lowest weight states (LWS's) or highest weight states (HWS's) of the η-spin and spin algebras.[18] In this paper we call these two algebras S_c and S_s *algebras*, respectively. In the case of the one-dimensional Hubbard model defined in the whole Hilbert space we find that the squares of the S_c and S_s operators—which for simplicity we denote $\hat{S}_c^2$ and $\hat{S}_s^2$, respectively—are independent conservation laws. We express these two conservation laws both in terms of electronic and pseudoparticle/pseudohole operators, with the final result that we find a total of $2N_a+4$ independent conservation laws.

Having established the number and nature of these conservation laws, we introduce an alternative set of independent pseudoparticle[4] conservation laws. We are able to show that these $2N_a+4$ conservation laws are independent and, because the expressions are considerably simpler—in terms of pseudoparticle operators—than those in the form derived from the transfer matrix, are much more useful for calculating physically relevant correlation functions for finite energy or frequency; indeed, results in this direction were or will be presented elsewhere.[5,7]

The remainder of the paper is organized as follows. In Sec. II we introduce some useful basic information both on the one-dimensional Hubbard model and on the pseudoparticle and Yang-particle algebra. The N_a+1 transfer-matrix currents, which are in one-to-one correspondence with the previously known N_a+1 transfer-matrix charges, are studied in Sec. III. Together with the $\hat{S}_c^2$ and $\hat{S}_s^2$ operators, these charges and currents constitute a set of $2N_a+4$ independent conservation laws. In Sec. IV we introduce an alternative set of $2N_a+4$ independent conservation laws with (relatively) simple expressions in terms of the pseudoparticle and Yang-particle operators. Finally, in Sec. V, we discuss our results and add some concluding remarks.

II. MODEL AND PSEUDOPARTICLE PICTURE

In a chemical potential μ and magnetic field H the Hubbard Hamiltonian reads

$$\hat{H}=\hat{H}_{SO(4)}+\sum_{\alpha}\mu_{\alpha}2\hat{S}_{\alpha}^{z}, \tag{1}$$

where α runs over c (for charge) and s (for spin). The Hamiltonian

$$\hat{H}_{SO(4)}=\hat{T}+U[\hat{D}-\hat{N}/2] \tag{2}$$

has SO(4) symmetry, and

$$\hat{T}=-t\sum_{j,\sigma}[c_{j\sigma}^{\dagger}c_{j+1\sigma}+\text{H.c.}],\quad \hat{D}=\sum_{j}\hat{n}_{j,\uparrow}\hat{n}_{j,\downarrow} \tag{3}$$

are the *kinetic-energy* and the double-occupancy operators, respectively. The operators

$$\hat{S}_c^z=-\tfrac{1}{2}[N_a-\hat{N}],\quad \hat{S}_s^z=-\tfrac{1}{2}[\hat{N}_{\uparrow}-\hat{N}_{\downarrow}], \tag{4}$$

are the diagonal generators of the SU(2) S_c and S_s algebras,[3] respectively. Using these standard definitions, we see that $\mu_c=\mu$ and $\mu_s=\mu_0 H$, with μ_0 the Bohr magneton and H the external magnetic field. We assume N_a is large and even, and recall, for definiteness, that the number operators are

$$\hat{N}=\sum_{\sigma}\hat{N}_{\sigma},\quad \hat{N}_{\sigma}=\sum_{j}\hat{n}_{j,\sigma},\quad \hat{n}_{j,\sigma}=c_{j\sigma}^{\dagger}c_{j\sigma}, \tag{5}$$

and $c_{j\sigma}^{\dagger}$ and $c_{j\sigma}$ are σ electron operators at site $j=1,\ldots,N_a$.

The Hamiltonian $\hat{H}_{SO(4)}$ [Eq. (2)] has SO(4) symmetry,[18] and commutes with the six generators of the S_c and S_s algebras, with the expressions of the two corresponding diagonal

generators being given in Eq. (4) and the off-diagonal generators of these two SU(2) algebras reading

$$\hat{S}_c^\dagger = \sum_j (-1)^j c_{j\downarrow}^\dagger c_{j\uparrow}^\dagger, \quad \hat{S}_c = \sum_j (-1)^j c_{j\uparrow} c_{j\downarrow} \tag{6}$$

and

$$\hat{S}_s^\dagger = \sum_j c_{j\downarrow}^\dagger c_{j\uparrow}, \quad \hat{S}_s = \sum_j c_{j\uparrow}^\dagger c_{j\downarrow}, \tag{7}$$

respectively. The operators $\hat{S}_c^2$ and $\hat{S}_s^2$ can be expressed in terms of the generators of Eqs. (4), (6), and (7) as follows:

$$\hat{S}_\alpha^2 \equiv \vec{\hat{S}}_\alpha . \vec{\hat{S}}_\alpha = \hat{S}_\alpha^\dagger \hat{S}_\alpha + [\hat{S}_\alpha^z]^2 - \hat{S}_\alpha^z . \tag{8}$$

In Eq. (8) and throughout this paper, we use units $\hbar = 1$. Moreover, we also use units with a lattice spacing $a = 1$ and an electron charge $-e = 1$. We consider densities $n = N/N_a$ and spin densities $m = [N_\uparrow - N_\downarrow]/N_a$ in the domains $0 \leqslant n \leqslant 1$ and $1 \leqslant n \leqslant 2$ and $-n \leqslant m \leqslant n$ and $-(2-n) \leqslant m \leqslant (2-n)$, respectively.

The Bethe ansatz solvability of the one-dimensional Hubbard model [Eq. (1)] is restricted to the Hilbert subspace spanned by the LWS's (Refs. 1 and 2) or HWS's (Ref. 17) of the S_c and S_s algebras, i.e., by the states whose S_α and S_α^z numbers are such that $S_\alpha = -S_\alpha^z$ or $S_\alpha = S_\alpha^z$, respectively[3]. The Bethe-ansatz solution used in this paper describes energy eigenstates associated with densities and spin densities in the domains $0 \leqslant n \leqslant 1$ and $0 \leqslant m \leqslant n$, respectively, i.e., associated with S_α LWS's. The description of the states corresponding to the extended n and m domains mentioned above is achieved by application of the off-diagonal generators of the SU(2) S_c and S_s algebras[18] on the LWS's, as we discuss below.

The use of the SO(4) algebra reveals that the model is integrable in the whole Hilbert space.[18] The transfer-matrix charges are closely related to that integrability, and the total number of pseudoparticle branches of the operator basis of Ref. 4 is $N_a + 1$, and is equal to the number of these independent conservation laws. These pseudoparticle branches are labeled by the pseudoparticle quantum numbers α and γ with $\alpha = c, s$ and $\gamma = 0,1,2,\ldots,N_a/2$ and $0,1,2,\ldots,N_a/2 - 1$ for c and s, respectively, which indeed gives a total number of $N_a + 1$ independent branches.

The α,γ pseudoparticles are well-defined objects when N_a is large, and the string hypothesis becomes valid.[2,3] Further, from the inverse-scattering Bethe-ansatz solution,[16,17] one can extract explicit expressions for the Hamiltonian, the other transfer-matrix charges (as we find in Sec. III), and the diagonal parts (in the energy basis) of all operators in terms of elementary anticommuting α,γ pseudoparticle operators $b_{q,\alpha,\gamma}^\dagger$ and $b_{q,\alpha,\gamma}$ such that[4]

$$\{b_{q,\alpha,\gamma}^\dagger, b_{q',\alpha',\gamma'}\} = \delta_{q,q'} \delta_{\alpha,\alpha'} \delta_{\gamma,\gamma'} . \tag{9}$$

At each α,γ band there are $N_{\alpha,\gamma}^* q_j$ values, i.e., $j = 1,2,\ldots,N_{\alpha,\gamma}^*$, with

$$q_{j+1} - q_j = \frac{2\pi}{N_a}. \tag{10}$$

This property of the pseudomomentum q_j correspond to the ideal positions of the Bethe-ansatz rapidities associated with the string hypothesis. Moreover,

$$q_{\alpha,\gamma}^{(-1)} \leqslant q \leqslant q_{\alpha,\gamma}^{(+1)}, \tag{11}$$

where the limits of the (α,γ)-band pseudo-Brillouin zones read

$$q_{\alpha,\gamma}^{(\iota)} = \iota q_{\alpha,\gamma} + \delta_{\alpha,c}\, \delta_{\gamma,0} y_{c,0} \frac{\pi}{N_a}, \tag{12}$$

$$q_{\alpha,\gamma} = \frac{\pi}{N_a}[N_{\alpha,\gamma}^* - 1], \quad y_{c,0} = \pm\frac{1}{2}[1 + (-1)^{N_0}] = 0, \pm 1; \tag{13}$$

the index

$$\iota = \mathrm{sgn}(q) 1 \tag{14}$$

refers to right $(\iota = +1)$ and left $(\iota = -1)$ α,γ pseudoparticle movers whose numbers, $N_{\alpha,\gamma,\iota}$, are independently conserved and are eigenvalues of the operators $\hat{N}_{\alpha,\gamma,\iota}$; $N_0 = (\Sigma_{\alpha,\gamma} N_{\alpha,\gamma} - N_{c,0})$; and $N_{\alpha,\gamma} = \Sigma_\iota N_{\alpha,\gamma,\iota}$ denotes the number of occupied q_j values at the α,γ band—the α,γ pseudoparticles.

As we shall show in Sec. III, the independent conservation of the $\iota = \pm 1$ pseudoparticle numbers plays a central role in the derivation of $N_a + 1$ transfer-matrix *currents* that go beyond the transfer-matrix charges found by Shastry. Along with the $N_{\alpha,\gamma}$ occupied pseudomomentum values, there are $N_{\alpha,\gamma}^h$ *empty* pseudomomentum values, which we call α,γ pseudoholes, in each α,γ band: we have

$$N_{\alpha,\gamma}^* = N_{\alpha,\gamma} + N_{\alpha,\gamma}^h,$$

$$N_{\alpha,\gamma}^h = N_{\alpha,0}^h - \sum_{\gamma'>0} [\gamma + \gamma' - |\gamma - \gamma'|] N_{\alpha,\gamma'} \tag{15}$$

and

$$N_{c,0}^h = N_a - N_{c,0}, \quad N_{s,0}^h = N_{c,0} - 2\sum_{\gamma'} N_{s,\gamma'} . \tag{16}$$

Equations (15) and (16) reveal that the numbers of α,γ pseudoholes, $N_{\alpha,\gamma}^h$, are fully determined by the set of α,γ pseudoparticle numbers $\{N_{\alpha,\gamma}\}$. Furthermore, the discrete pseudomomentum values are such that

$$q_j = \frac{2\pi}{N_a} I_j^{\alpha,\gamma}, \tag{17}$$

where $I_j^{c,0}$ (or $I_j^{\alpha,\gamma}$ for all remaining α,γ bands) are integers and half-integers for $N_a/2 + N_0$ (or $N_{\alpha,\gamma}^*$) odd and even, respectively.

In Ref. 4, two types of $\alpha,0$ pseudoholes were considered, which were distinguished by an extra quantum number, $\beta = \pm\frac{1}{2}$. In that reference the S_α non-LWS's were described in

terms of α,β pseudoholes (corresponding to the two types $\beta=\pm\frac{1}{2}$ of $\alpha,0$ pseudoholes) by anticommuting elementary operators $a^{\dagger}_{q,\alpha,\beta}$ and $a_{q,\alpha,\beta}$ and a number operator $\hat{N}^{h}_{\alpha,\beta}=\Sigma_q \hat{N}^{h}_{\alpha,\beta}(q)$ where $\hat{N}^{h}_{\alpha,\beta}(q)=a^{\dagger}_{q,\alpha,\beta}a_{q,\alpha,\beta}$. In terms of these α,β pseudohole operators, the SU(2) S_α generators of Eqs. (4), (6), and (7) read

$$\hat{S}^{z}_{\alpha}=\sum_{\beta=\pm 1/2}\beta\,\hat{N}^{h}_{\alpha,\beta},\quad \hat{S}^{\dagger}_{\alpha}=\sum_{q}\;a^{\dagger}_{q,\alpha,+(1/2)}a_{q,\alpha,-(1/2)},$$

$$\hat{S}_{\alpha}=\sum_{q}\;a^{\dagger}_{q,\alpha,-(1/2)}a_{q,\alpha,+(1/2)}, \tag{18}$$

and the numbers S_α associated with the eigenvalues $S_\alpha[S_\alpha+1]$ of the operators (8) can be written as

$$S_\alpha=\tfrac{1}{2}N^{h}_{\alpha,0}-\sum_{\gamma>0}\gamma N_{\alpha,\gamma},\quad N^{h}_{\alpha,0}=\sum_{\beta=\pm(1/2)}N^{h}_{\alpha,\beta}, \tag{19}$$

where the pseudohole numbers $N^{h}_{\alpha,0}$ can also be expressed in terms of pseudoparticle numbers—see Eq. (16).

For our later discussion, we shall find another representation of the non-LWS's to be useful. It involves describing the non-LWS's by applying onto the S_α LWS's of creation operators for *c* and *s Yang particles* having momenta $q_c=\pi$ and $q_s=0$, respectively. These operators are

$$d^{\dagger}_{q_\alpha}\equiv\frac{\hat{S}^{\dagger}_{\alpha}}{\sqrt{-2\hat{S}^{z}_{\alpha}}},\quad d_{q_\alpha}\equiv\frac{\hat{S}_{\alpha}}{\sqrt{-2\hat{S}^{z}_{\alpha}}}, \tag{20}$$

and obey *commutation* relations

$$[d_{q_\alpha},d^{\dagger}_{q_{\alpha'}}]=\delta_{\alpha,\alpha'}. \tag{21}$$

The S_α SU(2) generators $\hat{S}^{z}_{\alpha}$, $\hat{S}^{\dagger}_{\alpha}$, and $\hat{S}_{\alpha}$, involved in the expression of the α Yang-particle operators of Eq. (20), are expressed in Eqs. (4), (6), and (7) in terms of electronic operators, and in Eqs. (18) in terms of α,β pseudoholes.

III. TRANSFER-MATRIX CHARGES AND TRANSFER-MATRIX CURRENTS

For $U=0$, the $2N_a$ operators

$$\hat{N}_{\sigma}(k)=c^{\dagger}_{k\sigma}c_{k\sigma}, \tag{22}$$

and associated $2N_a$ operators

$$\hat{N}^{h}_{\sigma}(k)=c_{k\sigma}c^{\dagger}_{k\sigma}, \tag{23}$$

where $c^{\dagger}_{k\sigma}$ and $c_{k\sigma}$ are σ electron operators at momentum k with discrete values

$$k_j=\frac{2\pi}{N_a}j,\quad j=-\frac{N_a}{2}+1,-\frac{N_a}{2}+2,\ldots,0,\ldots,\frac{N_a}{2}-1,\frac{N_a}{2}, \tag{24}$$

commute with Hamiltonian (1) and are conservation laws. It is convenient to introduce the quantum number $\nu=\mathrm{sgn}(k)1$ for $k\neq 0$ which refers to the number of right-moving ($\nu=+1$) and left-moving ($\nu=-1$) *electrons*. At $U=0$ all conservation laws can be expressed in terms of the operator (22) [or (23)]. For instance, one can construct $2N_a$ local conservation laws, associated with two σ ($\uparrow$ and $\downarrow$) decoupled free-fermion models and $2N_a$ corresponding nonlocal laws, for the number right- and left-moving *electrons* are independent conservation laws. Furthermore, in contrast to the $U>0$ case, at $U=0$ one can rewrite *all* $4N_a$ conservation laws in a local form.[11]

On the other hand, for finite values of U the electronic operators (22) and (23) do not commute with Hamiltonian (1). However, all energy and momentum eigenstates *can* be described by distributions $N_{\alpha,\gamma}(q)$ and $N^{h}_{\alpha,\gamma}(q)$ for pseudoparticles and pseudoholes, respectively.[4] These distributions are the eigenvalues of the operators

$$\hat{N}_{\alpha,\gamma}(q)=b^{\dagger}_{q,\alpha,\gamma}b_{q,\alpha,\gamma} \tag{25}$$

and

$$\hat{N}^{h}_{\alpha,\gamma}(q)=b_{q,\alpha,\gamma}b^{\dagger}_{q,\alpha,\gamma}, \tag{26}$$

respectively. [Note that for $\gamma=0$ we have that $\hat{N}^{h}_{\alpha,0}(q)=\Sigma_{\beta=\pm(1/2)}\hat{N}^{h}_{\alpha,\beta}(q)$.] In addition, we consider the α Yang-particle operators

$$\hat{\mathcal{N}}_{\alpha}=d^{\dagger}_{q_\alpha}d_{q_\alpha}, \tag{27}$$

whose eigenvalues are the number of α Yang particles,

$$\mathcal{N}_\alpha=S^{\alpha}+S^{\alpha}_{z}, \tag{28}$$

such that $0\leqslant\mathcal{N}_\alpha\leqslant 2S_\alpha$ and with $\mathcal{N}_\alpha=0$ for the S_α LWS's. Operator (27) can be written in terms of α,β pseudohole (relative to the $\alpha,\gamma=0$ band) and α,γ pseudoparticle number operators as follows:

$$\hat{\mathcal{N}}_{\alpha}=\hat{N}^{h}_{\alpha,+(1/2)}-\sum_{\gamma>0}\gamma\hat{N}_{\alpha,\gamma}. \tag{29}$$

For finite values of U, the elementary q,α,γ pseudoparticle operators [Eq. (25)] and the α Yang-particle operators [Eq. (29)] commute with Hamiltonian (1), i.e., *these operators are conserved and thus represent conservation laws*. Note, however, that in contrast to the electronic operators [Eq. (22)], the number of q,α,γ pseudoparticle operators of Eq. (25) depends on the pseudoparticle fillings. To understand this point, we introduce the useful concept of a ''subcanonical ensemble Hilbert subspace'' (SEHS), which is spanned by all energy and momentum eigenstates with the same set of numbers $\{N_{\alpha,\gamma,\iota}\}$ and $\{\mathcal{N}_\alpha\}$. Unlike the electronic *operators* [Eq. (22)], whose number is $2N_a$ independently of the values of the electron numbers N_σ, the number of the operators [see Eq. (25)] is different for different SEHS's. This is because the pseudomomentum discrete values q_j of Eq. (17) are such that $j=1,2,\ldots,N^{*}_{\alpha,\gamma}$, where according to Eq. (15), and in contrast to a noninteracting system, the values of the numbers $N^{*}_{\alpha,\gamma}$ depend on the pseudoparticle occupancies. It follows that, unlike the $2N_a$ electronic operators [Eq. (22)], the operators of form (25) are not in general independent.

However, these operators have an important property: *All* conservation laws of the one-dimensional Hubbard model can be expressed in terms of the elementary operators (25) and (29), as we discuss below.

Our first goal in this paper is to identify the maximal set of *independent* conservation laws. By analogy to classical mechanics, the integrability of model (1) ought to require a *minimum* number of independent commuting conservation laws equal to the number of degrees of freedom, which is $2N_a$. Therefore, we expect that out of operators (25) and (29) one can construct *at least* $2N_a$ independent conservation laws.

The N_a+1 transfer-matrix charges include the operators $\hat{S}_c^z$ and $\hat{S}_s^z$ of Eq. (4), which are associated with the electron number and the magnetization, respectively. For the one-dimensional Hubbard model defined in the Hilbert subspace spanned by S_α LWS's, we have that $S_\alpha=-S_\alpha^z$, and the two $\hat{S}_\alpha^2$ operators of Eq. (8) and the two diagonal SU(2) S_α generators $\hat{S}_\alpha^z$ of Eq. (4) correspond to the same numbers.

On the other hand, for the one-dimensional Hubbard model defined in the whole Hilbert space, the two operators of Eq. (8) become independent conservation laws relative to the set of N_a+1 transfer-matrix charges, which gives a total number of N_a+3 independent conservation laws. The use of Eqs. (4), (6), (7), and (18) leads directly to expressions for the two conservation laws of Eq. (8) in the electronic and pseudoparticle/pseudohole basis, respectively. Therefore, we know how to write N_a+3 independent conservation laws in terms of electronic operators. These are the two operators of Eq. (8), and the previously known N_a+1 transfer-matrix charges of Refs. 8–11 and 17. This still leaves us short of our expected number of $\sim 2N_a$ conservation laws.

To find the remaining laws, we take a hint from the results for the case $U=0$. For $U=0$, the numbers of right- and left-moving electrons are *independently* conserved. Thus we can in principle immediately write down twice as many conservation laws; later, as an explicit check of our general results, we shall do this for $U/t\rightarrow 0$. For now, however, we proceed by observing that whereas for finite values of U the numbers of right- and left-moving electrons are no longer good quantum numbers, the numbers of right- and left-moving *pseudoparticles* associated with the quantum numbers of Eq. (14) *are* good quantum numbers. This simple but key insight underlies the detailed calculations that follow in the remainder of this section. As a guide to these calculations, let us briefly outline the logic behind them. If the numbers of right- and left-moving pseudoparticles are separately conserved, so must their sums and differences. By studying the expressions for known conservation laws corresponding to transfer-matrix charges in terms of the pseudoparticles, we will eventually be able to show that their expressions involve the *addition* of two separate sums over right- and left-moving pseudoparticle operators. This suggests that we look for the conservation laws to be expressed in terms of similar sums over the *differences* of right- and left-moving pseudoparticles. Doing so indeed allows us to find a new set of N_a+1 independent conservation laws which are *nonlocal* in the original electronic basis even in the limit $U/t\rightarrow 0$.

Let us start our detailed analysis by finding expressions for the N_a+1 known transfer-matrix charges in terms of pseudoparticle operators. The transfer-matrix charges with the simplest expressions are $\hat{S}_c^z$ and $\hat{S}_s^z$. In addition to their representations in terms of electrons, [Eq. (4)] and of α,β pseudoholes [Eq. (18)], we can write these diagonal generators of the S_c and S_s algebras in terms of the basic q,α,γ pseudoparticle operators [Eq. (25)] and α Yang-particle operators [Eq. (27)]. The corresponding expressions involve simple q and γ summations of the operators [Eq. (25)], and read

$$\hat{Q}^1=\hat{S}_c^z=-\tfrac{1}{2}[N_a-\hat{N}]=-\tfrac{1}{2}\sum_q\,[1-\hat{N}_{c,0}(q)]$$
$$+\sum_{\gamma>0}\sum_q\,\gamma\hat{N}_{c,\gamma}(q)+\hat{\mathcal{N}}_c,$$
$$\hat{Q}^2=\hat{S}_s^z=-\tfrac{1}{2}[\hat{N}_\uparrow-\hat{N}_\downarrow]=-\tfrac{1}{2}\sum_q\,\hat{N}_{c,0}(q)+\sum_q\,\hat{N}_{s,0}(q)$$
$$+\sum_{\gamma>0}\sum_q\,[1+\gamma]\hat{N}_{s,\gamma}(q)+\hat{\mathcal{N}}_s, \tag{30}$$

where we have defined the two transfer-matrix charges as $\hat{Q}^1$ and $\hat{Q}^2$, respectively. Moreover, we denote by $\hat{Q}^m$ with $m=1,2,\ldots,N_a+1$ the N_a+1 transfer-matrix charges of Refs. 8–11 and 17.

On the other hand, the expressions of all remainder N_a-1 transfer-matrix charges do not involve the α Yang-particle operators [Eq. (27)] and are functionals exclusively of the basic q,α,γ pseudoparticle operators [Eq. (25)]. However, their expressions in terms of the latter pseudoparticle operators are more involved than expressions (30). All these N_a-1 operators can be expressed in terms of the operators $\hat{N}_{c,\gamma}(q)$ and of the *rapidity* operators

$$\hat{R}_{c,0}(q)=4t\,\sin[\hat{K}(q)]/U,\quad \hat{R}_{c,\gamma}(q), \tag{31}$$

whose expressions are functionals of all the basic operators $\hat{N}_{\alpha,\gamma}(q)$ of Eq. (25). Therefore, rapidity operators (31) commute with Hamiltonian (1), and the energy eigenstastes are also eigenstates of these operators. Rapidity operators (31) play an important role in the pseudoparticle operator representation of the model. (For further information on these operators, see Sec. IV of Ref. 4.) The functional dependence of the eigenvalues of operators (31), $R_{c,0}(q)=4t\,\sin[K(q)]/U$ and $R_{c,\gamma}(q)$, on the distributions $N_{\alpha',\gamma'}(q')$ are defined by the coupled integral equations (66)–(69) of Ref. 4. As discussed in that reference, the latter equations are fully equivalent to the set of thermodynamic Bethe-ansatz equations first derived by Takahashi within his string hypothesis.[2] These eigenvalues are the real part of Bethe-anstaz rapidities[2–4] and are functionals of the pseudoparticle distributions $N_{\alpha',\gamma'}(q')$.

Solving the above coupled integral equations to find the expressions for the functionals $R_{c,\gamma}(q)$ in terms of the

pseudoparticle distributions is an involved problem. However, it is straightforward to show that that solution leads to expressions for these functionals that are summations over the quantum numbers q', α', and γ' involving the distributions $N_{\alpha',\gamma'}(q')$. The corresponding operators $\hat{R}_{c,\gamma}(q)$ have the same expressions, with the distributions $N_{\alpha',\gamma'}(q')$ replaced by the corresponding operators $\hat{N}_{\alpha',\gamma'}(q')$.

In order to show that the expressions of the $m>2$ transfer-matrix charges involve the rapidity operators of Eq. (31), we consider the auxiliary eigenvalue of Eq. (99) of Ref. 17. Although the authors of that reference considered the Bethe-ansatz solution associated with HWS's of the S_α algebras, it is straightforward to derive from their expressions the corresponding functions for the Bethe-ansatz solution associated with S_α LWS's. The main point is that all coefficients of the λ expansion of the logarithm of the auxiliary eigenvalue, [Eq. (106) of Ref. 17], only involve the rapidity function (107) of the same reference. In our case k_j is the discrete value of the rapidity function $K(q_j)$, which is the eigenvalue of the corresponding operator $\hat{K}(q_j)$ of Eq. (31).

By considering the limit of large N_a and introducing the rapidities associated with the string hypothesis, we find that all the $m>2$ transfer-matrix charges $\hat{Q}^m$ have the following general expression in the pseudoparticle basis:

$$\hat{Q}^m=\delta_{m,3}(U/4)N_a+\sum_q \hat{N}_{c,0}(q)\,\mathcal{O}_0^m[\hat{K}(q)] + \sum_{\gamma>0}\sum_q \hat{N}_{c,\gamma}(q)\,\mathcal{O}_\gamma^m[\hat{R}_{c,\gamma}(q)]. \tag{32}$$

Here $m=3,4,\ldots,N_a+1$, and the functions $\mathcal{O}_0^m(x)$ and $\mathcal{O}_\gamma^m(x)$ can be extracted from combining the inverse-scattering Bethe ansatz solution of Ref. 17 with the string hypothesis rapidities. General equation (32) confirms that the expression of the $m>2$ transfer-matrix charges only involve the operators $\hat{N}_{c,\gamma}(q)$ and the rapidity operators $\hat{K}(q)$ and $\hat{R}_{c,\gamma}(q)$ of Eq. (31). However, the charges [Eq. (32)] involve all operators $\hat{N}_{\alpha,\gamma}(q)$ of Eq. (25) because the rapidity operators are functionals of these basic operators.

To clarify this general result, we consider the explicit expression in the pseudoparticle basis of Hamiltonian (2), which is the transfer-matrix charge $\hat{Q}^3$, and the expression for the first nontrivial transfer-matrix charge commuting with it, which we denote by $\hat{Q}^4$. In terms of electronic operators this first nontrivial transfer-matrix charge reads[8,17]

$$\hat{Q}^4=-it\sum_{j,\sigma}[c_{j\sigma}^\dagger c_{j+2\sigma}-c_{j+2\sigma}^\dagger c_{j\sigma}] -iU\sum_{j,\sigma}[c_{j\sigma}^\dagger c_{j+1\sigma}-c_{j+1\sigma}^\dagger c_{j\sigma}][\hat{n}_{j+1,\sigma}+\hat{n}_{j,\sigma}-1]. \tag{33}$$

Recomputing expansion (106) of Ref. 17 for electrons instead of holes, introducing the $\gamma>0$ rapidities associated with the string hypothesis, and using the pseudoparticle representation introduced in Ref. 4, after some algebra we arrive at the following expressions for Hamiltonian (2) and the first nontrivial transfer-matrix charge (33) in terms of the pseudoparticle operators

$$\begin{aligned}\hat{Q}^3=\hat{H}_{SO(4)}=&-2t\sum_q \hat{N}_{c,0}(q)\,\{\cos[\hat{K}(q)]+(U/8t)\}\\ &+(U/4)\sum_q[1-\hat{N}_{c,0}(q)]\\ &-\sum_{\gamma>0}\sum_q \hat{N}_{c,\gamma}(q)\Big\{\gamma U-2t\\ &\times\sum_{l=\pm1}\sqrt{1-(U/4t)^2[\hat{R}_{c,\gamma}(q)+il\gamma]^2}\Big\}\end{aligned} \tag{34}$$

and

$$\begin{aligned}\hat{Q}^4=&4t\sum_q \hat{N}_{c,0}(q)\,\{\cos[\hat{K}(q)]+(U/2t)\}\sin[\hat{K}(q)]\\ &+U\sum_{\gamma>0}\sum_q \hat{N}_{c,\gamma}(q)\Big\{(U/4t)(\gamma-1)\,\hat{R}_{c,\gamma}(q)\\ &-\sum_{l=\pm1}[\hat{R}_{c,\gamma}(q)+il\gamma]\\ &\times\sqrt{1-(U/4t)^2[\hat{R}_{c,\gamma}(q)+il\gamma]^2}\Big\},\end{aligned} \tag{35}$$

respectively. Comparision of general expression (32) with expressions (34) and (35) leads to

$$\mathcal{O}_0^3(x)=-2t\cos x-(U/2), \tag{36}$$

$$\mathcal{O}_\gamma^3(x)=-\gamma U+2t\sum_{l=\pm1}\sqrt{1-(U/4t)^2[x+il\gamma]^2},$$

and

$$\mathcal{O}_0^4(x)=4t\,[\cos x+(U/2t)]\sin x, \tag{37}$$

$$\mathcal{O}_\gamma^4(x)=U\Big\{(U/4t)(\gamma-1)\,x-\sum_{l=\pm1}[x+il\gamma]\times\sqrt{1-(U/4t)^2[x+il\gamma]^2}\Big\},$$

respectively. The same procedure leads to expressions for the remainder N_a-3 transfer-matrix charges, $\hat{Q}^m$, with $m=5,\ldots N_a+1$, in terms of pseudoparticle operators. Here omit the expressions for $m>4$, which are of the general form given in Eq. (32), with the functions $\mathcal{O}_0^m(x)$ and $\mathcal{O}_\gamma^m(x)$ more involved than expressions (36) and (37).

In addition to the conservation laws associated with these N_a-1 transfer-matrix charges, there are only two more in-

dependent local laws. These are associated with the SU(2) S_c and S_s algebras, and are the two operators $\hat{S}^z_\alpha$ (4). Furthermore, we know that the two *nonlocal* operators, $\hat{S}^2_\alpha$ (8) also lead to independent conservation laws. We thus have identified N_a+3 independent conservation laws. Armed with these results, we shall next show that the conservation laws corresponding to the transfer-matrix charges can be expressed as sums over separately conserved contributions of right- and left-moving pseudoparticles. Then, by looking at the difference of the right- and left-moving pseudoparticles, we shall see that this approach indeed leads to the expected independent conservation laws, which we shall call transfer-matrix currents and which are *nonlocal* in character, an attribute that explains their elusiveness.

We use the subscript ι introduced in Eq. (14) to distinguish right ($\iota=+1$) and left ($\iota=-1$) pseudoparticles and the corresponding operators. When expressed in terms of pseudoparticle operators, the N_a+1 transfer-matrix charges Q^m can be rewritten as

$$\hat{Q}^m=\sum_\iota \hat{Q}^m_\iota, \tag{38}$$

where, for $m=1$ and 2, we find

$$\hat{Q}^1_\iota=-\tfrac{1}{2}\sum_q \Theta(\iota q)[1-\hat{N}_{c,0}(q)] + \sum_{\gamma>0}\sum_q \Theta(\iota q)\gamma\hat{N}_{c,\gamma}(q)+\tfrac{1}{2}\hat{\mathcal{N}}_c, \tag{39}$$

$$\hat{Q}^2_\iota=-\tfrac{1}{2}\sum_q \Theta(\iota q)\hat{N}_{c,0}(q)+\sum_q \Theta(\iota q)\hat{N}_{s,0}(q) + \sum_{\gamma>0}\sum_q \Theta(\iota q)[1+\gamma]\hat{N}_{s,\gamma}(q)+\tfrac{1}{2}\hat{\mathcal{N}}_s,$$

and for $m>2$ operators (38) read

$$\hat{Q}^m_\iota=\delta_{m,3}(U/8)N_a+\sum_q \Theta(\iota q)\hat{N}_{c,0}(q)\,\mathcal{O}^m_0[\hat{K}(q)] + \sum_{\gamma>0}\sum_q \Theta(\iota q)\hat{N}_{c,\gamma}(q)\,\mathcal{O}^m_\gamma[\hat{R}_{c,\gamma}(q)], \tag{40}$$

and $\Theta(\iota q)=1$ for $\iota q>0$, $\Theta(0)=1/2$, and $\Theta(\iota q)=0$ for $\iota q<0$. Thus, as anticipated, the conservation laws corresponding to transfer-matrix charges can be expressed in terms of ''right plus left'' pseudoparticle operators.

Since the operators $\hat{Q}^m_{-1}$ and $\hat{Q}^m_{+1}$ are conserved independently, we can define immediately the following set of associated N_a+1 operators, which in anticipation of our results we shall denote as transfer-matrix currents $\hat{J}^m$,

$$\hat{J}^m\equiv\tfrac{1}{2}\sum_\iota \iota\hat{Q}^m_\iota, \tag{41}$$

which, for $m=1$ and 2 can be written as

$$\hat{J}^1=\frac{1}{2}\left\{\sum_\iota \frac{1}{2}\sum_q \Theta(\iota q)\iota\hat{N}_{c,0}(q) + \sum_{\gamma>0}\sum_\iota\sum_q \Theta(\iota q)\iota\gamma\hat{N}_{c,\gamma}(q)\right\}, \tag{42}$$

$$\hat{J}^2=\frac{1}{2}\left\{-\frac{1}{2}\sum_\iota\sum_q \Theta(\iota q)\iota\hat{N}_{c,0}(q)+\sum_\iota\sum_q \Theta(\iota q)\iota\hat{N}_{s,0}(q) + \sum_{\gamma>0}\sum_\iota\sum_q \Theta(\iota q)\iota[1+\gamma]\hat{N}_{s,\gamma}(q)\right\},$$

and for $m>2$ read

$$\hat{J}^m=\frac{1}{2}\left\{\sum_\iota\sum_q \Theta(\iota q)\iota\hat{N}_{c,0}(q)\,\mathcal{O}^m_0[\hat{K}(q)] + \sum_{\gamma>0}\sum_\iota\sum_q \Theta(\iota q)\iota\hat{N}_{c,\gamma}(q)\,\mathcal{O}^m_\gamma[\hat{R}_{c,\gamma}(q)]\right\}. \tag{43}$$

That this set of N_a+1 operators [Eqs. (41)–(43)] corresponds to independent conservation laws follows immediately from their construction as the difference between separately and independently conserved right and left pseudoparticle operators.

Can we express these new conservation laws explicitly in terms of electron operators? Unfortunately, for general U, the answer is ''not yet.'' In the limit $U/t\to 0$, however, we have been able to find such expressions. That they are both nonlocal and nontrivial suggests that finding the results for general U will be a challenging problem.

For $U/t\to 0$, the left- and right-moving *electron operators* become separately conserved, and we can apply the same approach used above—taking the difference of right minus left instead of the sum—to generate the transfer matrix ''current'' operators associated with each transfer-matrix ''charge'' operator. The validity of this procedure is justified by the fact that in the limit of $U/t\to 0$ the expressions of right (left) electronic operators in terms of pseudoparticles involve only right (left) pseudoparticle operators. Starting with the expressions for the currents in terms of electron operators $c^\dagger_{k\sigma}$ and $c_{k\sigma}$ of momentum k, and using Fourier transforms to re-express these operators in terms of the electron operators in coordinate space, $c^\dagger_{j\sigma}$ and $c_{j\sigma}$, after some algebra we find expressions for the first few ($m=1$, 2, 3, and 4) transfer-matrix currents $\hat{J}^m$ associated with the transfer-matrix charges of Eqs. (30), (34), and (35). The results are

$$\hat{J}^1=\frac{i}{4\pi}\sum_{j\neq j',\sigma}\frac{[1-(-1)^{|j-j'|}]}{(j-j')}c^\dagger_{j\sigma}c_{j'\sigma}, \tag{44}$$

$$\hat{J}^2=\frac{i}{4\pi}\sum_{j\neq j'}\frac{[1-(-1)^{|j-j'|}]}{(j-j')}[c^\dagger_{j\uparrow}c_{j'\uparrow}-c^\dagger_{j\downarrow}c_{j'\downarrow}], \tag{45}$$

$$\hat{J}^3 = -\frac{it}{2\pi}\sum_{j\neq j',\sigma}[1+(-1)^{|j-j'|}]\sum_{l=\pm 1}\frac{1}{(j-j'+l)}c^{\dagger}_{j\sigma}c_{j'\sigma}, \tag{46}$$

and

$$\hat{J}^4 = \frac{it}{2\pi}\sum_{j\neq j',\sigma}[1-(-1)^{|j-j'|}]\sum_{l=\pm 1}\frac{l}{(j-j'+2l)}c^{\dagger}_{j\sigma}c_{j'\sigma}. \tag{47}$$

Applying similar manipulations to the higher transfer-matrix charges [Eq. (32)], we can find similar expressions for the remainder N_a-3 transfer-matrix currents [Eq. (41)]. Expressions (44)–(45) confirm the non-local character of the operators of Eq. (41): that is, in contrast to the expressions for the (local) transfer-matrix charges, which involve a single j summation, the expressions of operators (44)–(47) involve two j summations, i.e., are nonlocal. This nonlocal character also characterizes the transfer-matrix currents of Eq. (41) for finite values of U.

To summarize our results in this section, we have shown that the one-dimensional Hubbard model has N_a+1 local independent conservation laws and N_a+3 nonlocal independent conservation laws. In addition, we have provided explicit expressions for these operators in terms of pseudoparticle operators. The total number of the above independent conservation laws is $2N_a+4$, and is larger than the minimum number of $2N_a$ independent conservation laws required by the integrability of the model. When restricted to the Hilbert subspace associated with the Bethe ansatz solution, which is spanned by the S_α LWS's, the number of independent conservation laws is reduced to $2N_a+2$ for $S_\alpha=-S^z_\alpha$.

IV. PSEUDOPARTICLE AND YANG-PARTICLE CONSERVATION LAWS

Our results thus far have confirmed the expectation, based on the integrability of model (1), that there should be $\sim 2N_a$ independent conservation laws for that model defined on N_a lattice sites. The particular representation of these conservation laws follows from an extension of the earlier transfer-matrix analyses of the problem[8–11,17] and is not particularly simple when expressed in terms of either the pseudoparticles or the electrons; indeed, we do not have a general expression for the newly found nonlocal laws in terms of electron operators.

In this section we shall introduce an alternative set of independent conservation laws. Although we do not provide an explicit representation of most of these laws in terms of the original electron operators, this alternative choice has the considerable virtue of providing much simpler expressions in terms of the pseudoparticles, where it arises from N_a+1 pseudoparticle ''charge'' conservation laws and N_a+1 pseudoparticle ''current'' conservation laws which, together with the two Yang-particle conservation laws [Eq. (27)] yield a full set of $2N_a+4$ commuting, compatible operators. Importantly, as is established elsewhere,[5,7] the representation of the pseudoparticle conservation laws is better suited for study of correlation functions at finite energy or frequency than operators (32) and (41). Finally, the physics contained in the commuting pseudoparticle conservation laws can be further clarified by the study of a relation that exists between these laws and a hidden non-Abelian algebra which characterizes the model in the thermodynamic limit.[6]

For finite values of U one can construct N_a+1 pseudoparticle ''charge'' conservation laws

$$\hat{N}_{\alpha,\gamma} = \sum_q \hat{N}_{\alpha,\gamma}(q), \tag{48}$$

and N_a+1 ''current'' pseudoparticle laws

$$\hat{J}_{\alpha,\gamma} = \tfrac{1}{2}\sum_\iota\sum_q \Theta(\iota q)\,\iota\,\hat{N}_{\alpha,\gamma}(q). \tag{49}$$

Based on both the anticommuting α,γ pseudoparticle characters and the commuting α Yang particle characters, and on the completeness of the corresponding basis, we find that the set of N_a+3 commuting *charge* operators $\hat{\mathcal{N}}_\alpha$ of Eq. (27) and $\hat{N}_{\alpha,\gamma}$ of Eq. (48) correspond to independent conservation laws.

Moreover, since the α,γ pseudoparticles obey independent right ($\iota=+1$) and left ($\iota=-1$) conservation laws, the set of $2N_a+2$ operators $\hat{N}_{\alpha,\gamma,\iota}$ are independent conservation laws. It follows that the N_a+1 *current* operators $\hat{J}_{\alpha,\gamma}$ of Eq. (49) are also independent conservation laws.

As a result, the set of $2N_a+4$ charge and current operators, which includes the two charge Yang particle operators $\hat{\mathcal{N}}_\alpha$ [Eq. (27)], the N_a+1 charge pseudoparticle operators [Eq. (48)], and the N_a+1 current pseudoparticle operators [Eq. (49)] refer to independent conservation laws and commute with the Hamiltonian and among themselves, i.e.,

$$[\hat{H}_{SO(4)},\hat{\mathcal{N}}_\alpha]=[\hat{H}_{SO(4)},\hat{N}_{\alpha,\gamma}]=[\hat{H}_{SO(4)},\hat{J}_{\alpha,\gamma}]=0, \tag{50}$$

and also

$$[\hat{N}_{\alpha,\gamma},\hat{N}_{\alpha',\gamma'}]=[\hat{J}_{\alpha,\gamma},\hat{J}_{\alpha',\gamma'}]=[\hat{N}_{\alpha,\gamma},\hat{J}_{\alpha',\gamma'}]=0, \tag{51}$$

and $\hat{\mathcal{N}}_\alpha$ also commutes with all these operators. Thus we have just confirmed explicitly by a different method the result of Sec. III that the number of independent conservation laws of the one-dimensional Hubbard model is $2N_a+4$, and is indeed greater than the $2N_a$ needed to ensure integrability.

The two Yang-particle operators [Eq. (27)] are conservation laws for the one-dimensional Hubbard model defined in the whole Hilbert space. On the other hand, the Bethe-ansatz solution[1,2,16,17] refers to that model defined in the Hilbert subspace spanned by the S_α LWS's (or HWS's). In this case there are no α Yang particles, and the set of $2N_a+4$ conservations laws reduces to the set of $2N_a+2$ laws [Eqs. (48) and (49)].

Since all the momentum and energy eigenstates belonging the same SEHS have the same values for the conservation-law numbers $\{\mathcal{N}_\alpha\}$, $\{N_{\alpha,\gamma}\}$, and $\{J_{\alpha,\gamma}\}$, we often use these numbers to label these states, which we call

$|\psi_i;\{\mathcal{N}_\alpha\},\{N_{\alpha,\gamma}\},\{J_{\alpha,\gamma}\}\rangle$ (and sometimes just $|\psi_i\rangle$). However, this notation is not a complete representation for these states, for states belonging the same SEHS have different q occupancies. In the complete q,α,γ pseudoparticle basis and α Yang-particle basis, the momentum and energy eigenstates have a noninteracting form that reads

$$|\psi_i\rangle=\prod_\alpha \frac{1}{\mathcal{N}_\alpha!}[d^+_{q_\alpha}]^{\mathcal{N}_\alpha}|\psi_i;0\rangle, \quad |\psi_i;0\rangle=\prod_{q,\alpha,\gamma} b^\dagger_{q,\alpha,\gamma}|0\rangle, \tag{52}$$

where

$$|\psi_i;0\rangle=|\psi_i;0,\{N_{\alpha,\gamma}\},\{J_{\alpha,\gamma}\}\rangle \tag{53}$$

is a S_α LWS and $|0\rangle$ is the $N=0$ vacuum, and when $\mathcal{N}_\alpha=0$ we define $[d^\dagger_{q_\alpha}]^0\equiv 1$ and $0!\equiv 1$.

In contrast to Eq. (23) of Ref. 4, which constructs the energy eigenstates from the α,β pseudohole and $\alpha,\gamma>0$ pseudoparticle vacuum, states (52) and (53) are obtained by application of pseudoparticle and Yang-particle creation operators on the $N=0$ vacuum. Note that the former vacuum is the ground state of model (1) at $S^z_c=S^z_s=0$. This difference is a result of the use of α Yang particles instead of α,β pseudoholes in the description of the S_α non-LWS's.

Equation (52) reveals that in the pseudoparticle and Yang-particle bases, all energy eigenstates are products of simple Slater-determinant levels of α,γ pseudoparticles and of α Yang-particle creation operators. Note, however, that unlike a noninteracting system the integer or half-integer character of the discrete pseudomomentum numbers $I^{\alpha,\gamma}_j$ of Eq. (17) depends on the parities of the pseudoparticle occupancy numbers. This implies that the generators that map an energy eigenstate with finite pseudoparticle occupancy onto another energy eigenstate also with pseudoparticle occupancy are not as simple as the generators of expressions (52). While the latter generators are simple products of Yang-particle and pseudoparticle creation operators, the former generators involve in addition to Yang-particle and pseudoparticle creation and/or annihilation operators, α,γ pseudoparticle topological momentum-shift operators. The latter operators generate collective pseudoparticle excitations that shift by $\pm\pi/N_a$ the pseudomomenta of all α,γ pseudoparticles of the final state whose numbers $I^{\alpha,\gamma}_j$ of Eq. (17) change their integer or half-integer character.[4]

Moreover, despite the seemingly non-interacting pseudoparticle form of the energy eigenstates [Eq. (52)], the pseudoparticles are indeed interacting objects, although the integrability of the model implies that their collisions are dissipationless, i.e., do not lead to energy or momentum transfer and only give rise to shifts in the pseudoparticle phases.[4]

One can find alternative choices for the independent $2N_a+4$ conservation laws of Eqs. (27) and Eqs. (48) and (49). For instance, we can use the set of N_a+1 α,γ pseudohole operators $\hat{N}^h_{\alpha,\gamma}$ associated with the numbers $N^h_{\alpha,\gamma}$ of Eq. (15). Following Eqs. (15) and (16), the set of N_a+1 laws can be expressed in terms of the N_a+1 pseudoparticle conservation laws of Eq. (48), and contains the same information. Note, however, that there is not a one-to-one correspondence between these two set of alternative N_a+1 operators. Instead of the two operators of Eq. (27) we could also use the two $\alpha,\beta=+\frac{1}{2}$ pseudohole operators $\hat{N}^h_{\alpha,+1/2}$ of Eq. (29).

The two $\hat{S}^2_\alpha$ operators of Eq. (8), the N_a+1 transfer-matrix charges [Eq. (38)] of Refs. 8–11 and 17, and the N_a+1 transfer-matrix currents [Eq. (41)] are also alternative to the two Yang-particle operators [Eq. (27)], the N_a+1 charge pseudoparticle operators [Eq. (48)], and the N_a+1 current pseudoparticle operators [Eq. (49)]. We showed in Sec. III that the transfer-matrix charges and transfer-matrix currents can also be expressed in terms of the elementary operators of Eq. (25). However, their expressions are considerably more involved than the simple expressions (48) and (49).

Although the N_a+1 transfer-matrix charges [Eq. (38)] and N_a+1 transfer-matrix currents [Eq. (41)] are independent, there is not an obvious one-to-one correspondence between these operators and the N_a+1 charge pseudoparticle operators [Eq. (48)] and N_a+1 current pseudoparticles operators [Eq. (49)], respectively, as confirmed by comparing their expressions in terms of basic pseudoparticle operators. For instance, this can be directly confirmed for the charge operators $\hat{Q}^1$ and $\hat{Q}^2$ of Eq. (30) and current operators $\hat{J}^1$ and $\hat{J}^2$ of Eq. (42), the combined expressions of these four conservation laws involving *all* $2N_a+4$ Yang-particle and pseudoparticle operators (27), (48), and (49), and reading

$$\hat{Q}^1=-\tfrac{1}{2}[N_a-\hat{N}_{c,0}]+\sum_{\gamma>0}\gamma\hat{N}_{c,\gamma}+\hat{\mathcal{N}}_c,$$

$$\hat{Q}^2=-\tfrac{1}{2}\hat{N}_{c,0}+\hat{N}_{s,0}+\sum_{\gamma>0}[1+\gamma]\hat{N}_{s,\gamma}+\hat{\mathcal{N}}_s, \tag{54}$$

and

$$\hat{J}^1=\tfrac{1}{2}\hat{J}_{c,0}+\sum_{\gamma>0}\gamma\hat{J}_{c,\gamma},$$

$$\hat{J}^2=-\tfrac{1}{2}\hat{J}_{c,0}+\hat{J}_{s,0}+\sum_{\gamma>0}[1+\gamma]\hat{J}_{s,\gamma}. \tag{55}$$

Note that the expression of the $m=1$ (and $m=2$) transfer-matrix charges, [Eq. (54)], involves the c Yang-particle and all the c,γ charge pseudoparticle operators [the s Yang particle, the $c,0$ charge pseudoparticle, and all the s,γ charge pseudoparticle operators] [Eqs. (27) and (48)], whereas the expression of the $m=1$ [and $m=2$] transfer-matrix currents, Eq. (55), involves all the c,γ current pseudoparticle operators [the $c,0$ current pseudoparticle operator and all the s,γ current pseudoparticle operators] [Eq. (49)]. Since the transfer-matrix currents [Eq. (42)] are nonlocal operators when expressed in terms of electrons, Eq. (55) suggests that the N_a+1 current pseudoparticle conservation laws [Eq. (49)] are also nonlocal operators in the electron basis.

Equation (54) reveals that the expression of the two transfer-matrix charges $\hat{J}^1$ and $\hat{J}^2$ involve both the Yang-

particle operators [Eq. (27)] and the charge pseudoparticle operators [Eq. (48)]. While the two Yang-particle operators [Eq. (27)] are not independent conservation laws relatively to the set of N_a+1 transfer-matrix charges [Eq. (38)], they are independent operators relatively to the N_a+1 charge pseudoparticle operators [Eq. (48)]. On the other hand, the two $\hat{S}_\alpha^2$ operators [Eq.(8)] are independent relative to the set of N_a+1 transfer-matrix charges [Eq. (38)] but are not independent relative to the N_a+1 charge pseudoparticle operators [Eq. (48)]. However, our results show that one can always choose a set of $2N_a+4$ independent conservation laws, which is reduced to $2N_a+2$ laws for the Hilbert subspace associated with the Bethe-ansatz solvability of the one-dimensional Hubbard model [Eq. (1)].

V. DISCUSSION AND CONCLUDING REMARKS

The three main results of this paper are (i) the identification of a set of N_a+1 , independent, nonlocal ''transfer-matrix current'' conservation laws, which coupled with the previously known set of N_a+1 transfer-matrix ''charge'' conservation laws,[8–11,17] provide the $\sim 2N_a$ conservation laws that are expected by the integrability of the one-dimensional Hubbard model; (ii) the explicit expression of both the transfer-matrix charges and currents in terms of pseudoparticle operators, and the explicit form of the transfer-matrix currents in terms of electron operators for the special case $U/t\rightarrow 0$; and (iii) the derivation of an alternative complete set of conservation laws that have much simpler expressions in terms of the pseudoparticle operators, and are hence more useful for calculating physically relevant correlation functions.

In a sense the ''naturalness'' of the pseudoaprticle operators is what allowed us to find this full solution. While all energy eigenstates of the Hubbard model [Eq. (1)] can be constructed by application onto the $N=0$ vacuum of suitable generators expressed in terms of either electrons or pseudoparticles, the evaluation of expressions for these generators in terms of electronic operators is a very complex and open problem, whereas from Eqs. (52) and (53) we see that these generators can be expressed as simple products of α,γ pseudoparticle creation operators and α-Yang particle creation operators. In this sense, the pseudoparticle and Yang-particle representation naturally diagonalizees the quantum problem.

In order to construct the new N_a+1 transfer-matrix currents [Eq. (41)], we have expressed the associated N_a+1 transfer-matrix charges of Refs. 8–11 and 17 in the pseudoparticle basis. This was achieved by combining the expressions for these charges in terms of the Bethe-ansatz rapidities provided by the transfer matrix[17] with the rapidities of the string hypothesis.[2,4] Taking into account that the numbers of right and left pseudoparticles are independently conserved, we constructed the extra set of N_a+1 independent currents [Eq. (41)], beyond the transfer-matrix charges, which have a one-to-one correspondence with these charges.

In contrast to the independent conservation of the numbers of right and left pseudoparticle for all finite values of the on-site repulsion U, the numbers of right and left electrons become good quantum numbers only as $U/t\rightarrow 0$. It follows that the seemingly simple expressions [Eq. (41)] of the transfer-matrix currents in terms of pseudoparticle operators are likely to be complicated in terms of electron operators, and indeed expressions for a general U are unknown. We were able to find explicit expressions in terms of electron operators in the specific limit $U/t\rightarrow 0$. Both in this limit and for finite values of the on-site repulsion U, the transfer-matrix currents are nonlocal when expressed in terms of the electron operators. Their nonlocality explains their previous elusiveness.

The transfer-matrix charges and our transfer-matrix currents can be written in the pseudoparticle basis in terms of the basic q,α,γ pseudoparticle operators of Eq. (25), but their expressions involve the rapidity operators (31), which are functionals of these basic operators. These leads to fairly complex expressions. On the other hand, the expressions of the alternative set of pseudoparticle conservation laws [Eqs. (48) and (49)] are much simpler, referring directly to the numbers of pseudoparticles. This simplicity proves to be useful in applications which combine the symmetries[7] associated with the $2N_a+2$ pseudoparticle conservation laws [Eqs. (48) and (49)] and two Yang-particle conservation laws [Eq. (27)] with the nonlinear critical theory of Ref. 19. This leads to useful finite-energy expressions for correlation functions. A preliminary but nontrivial example of application of this finite-energy theory is the evaluation of the optical conductivity of the Hubbard model [Eq. (1)] for finite frequencies just above the optical gap.[5]

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