"Full Counting Statistics" in and out of equilibrium

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Characterizing Quantum Many-Body Systems

Complete description given by many-body state/wave-function

$$\Psi(x_1, x_2, \ldots, x_N, t) = \langle x_1, \ldots, x_N | \Psi(t) \rangle$$

In practice: typically measure particular expectation values

$$\langle \Psi | \mathcal{O}(x,t) | \Psi \rangle \qquad \langle \Psi | \mathcal{O}_1(x,t) \mathcal{O}(x',t') | \Psi \rangle$$

These correspond to **averages** over many measurements.

A lot more info in the **probability distribution** of a given observable $\mathcal{O}$ in a QM state $|\Psi\rangle$

$$P_{\mathcal{O}}(m) = \langle\Psi|\delta(\mathcal{O}-m)|\Psi\rangle = \sum_n |\langle n|\Psi\rangle|^2\delta(\lambda_n - m)$$

$$\mathcal{O}|n\rangle = \lambda_n|n\rangle$$

Why should we care about these quantities?

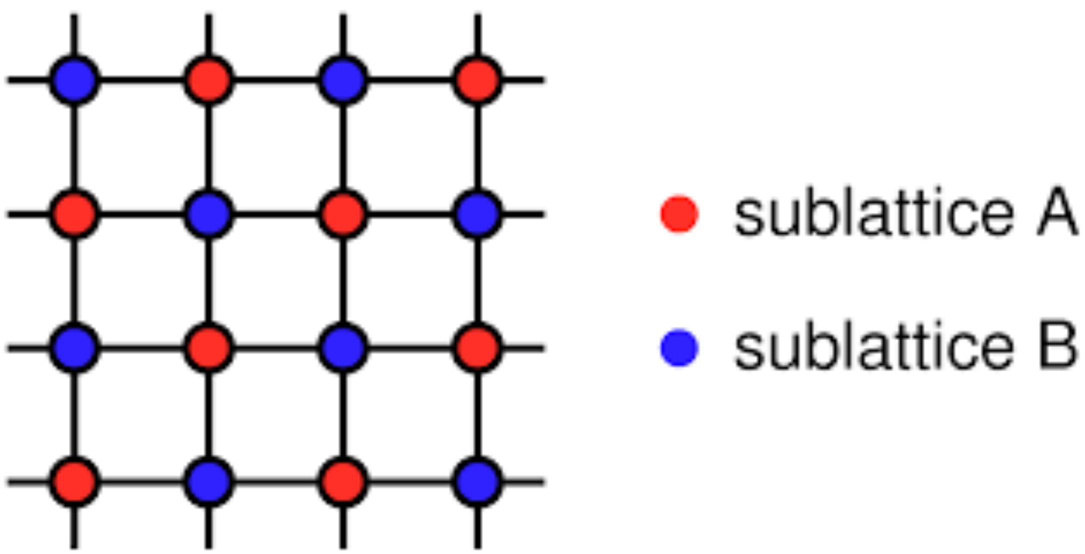
- They are measured in cold atom experiments
- They can be interesting (multiple peaks etc)

Cold Atom Experiments

probability distribution of staggered magnetization for 2D Hubbard model at finite temperature

Greiner group '17

$$H = -t \sum_{\langle j,k\rangle,\sigma=\uparrow,\downarrow} c^\dagger_{j,\sigma} c_{k,\sigma} + \text{h.c.} + U \sum_j c^\dagger_{j,\uparrow} c_{j,\uparrow} c^\dagger_{j,\downarrow} c_{j,\downarrow}$$



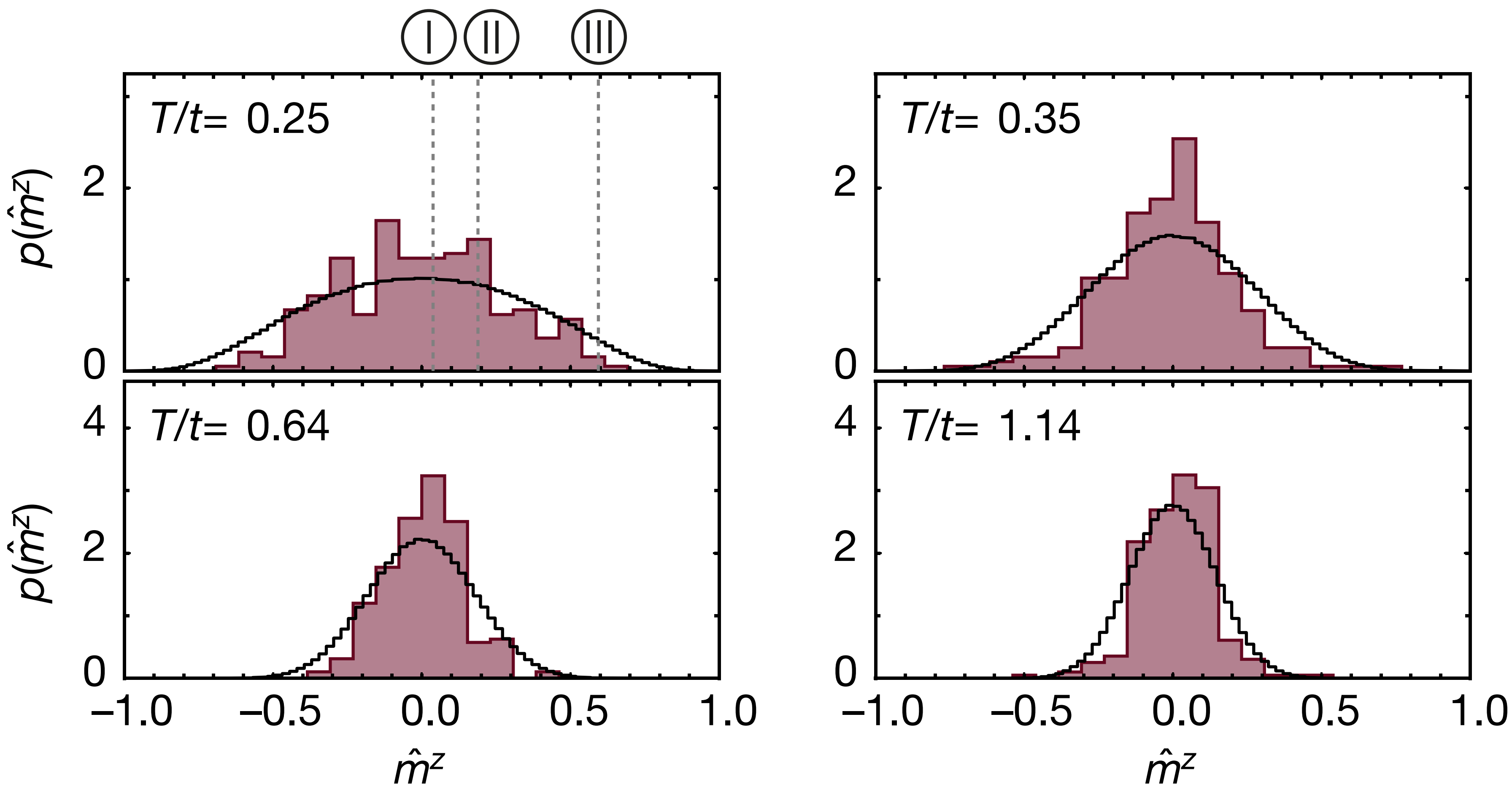
$$\hat{m}^z = \frac{2}{N^2}\left[\sum_{j\in A} S^z_j - \sum_{j\in B} S^z_j\right]$$

$$2S^z_j = c^\dagger_{j,\uparrow} c_{j,\uparrow} - c^\dagger_{j,\downarrow} c_{j,\downarrow}$$

- Initialize system in some initial state
- Measure m^z
- Repeat

probability distribution of staggered magnetization for 2D Hubbard at finite temperature

Greiner group '17



Other examples :

Probability distribution of "relative phase" in split 1D Bose gases

Schmiedmayer group
'10–'17

Very few results available in the literature for such quantities:

- $\int_0^\ell dx\, e^{i\Phi(x)}$ in Luttinger liquid

  Gritsev/Altman/Demler/Polkovnikov '06
  Kitagawa et al '10

- total magnetization in GS of critical Ising QFT

  Lamacraft/Fendley '08

- some numerics for GS of XXZ

  Moreno-Cardoner et al '16

- GS of Haldane-Shastry

  Stephan/Pollmann '17

- total transverse magnetisation in GS of TFIM & related free fermion problems

  Cherng & Demler '07
  Ivamov&Abanov'13
  Klich '14...

Other models/observables?

Have looked at the following cases:

- sub-system magnetisations in ground state of the critical spin-1/2 Heisenberg XXZ chain; Collura, FHLE &Groha ‘17
- transverse sub-system magnetization in TFIM in equilibrium and after quantum quenches; Groha, FHLE &Calabrese ‘18
- order parameter after a Néel quench in the Heisenberg XXZ chain Collura & FHLE
- order parameters in ground state of 1D Hubbard model FHLE &Vernier

Some generalities

Consider

- lattice models
- observables $\mathcal{O}$ (quantised eigenvalues) that act on sub-systems of linear size ℓ, e.g. sub-system magnetisation.

In states with finite correlation length ξ and $\xi \ll \ell$ we expect

$$P_{\mathcal{O}}(m) = \langle \Psi | \delta(\mathcal{O} - m) | \Psi \rangle = \sum_r P_w(r) \delta(m - r)$$

narrow, ≈ Gaussian

("thermodynamics")

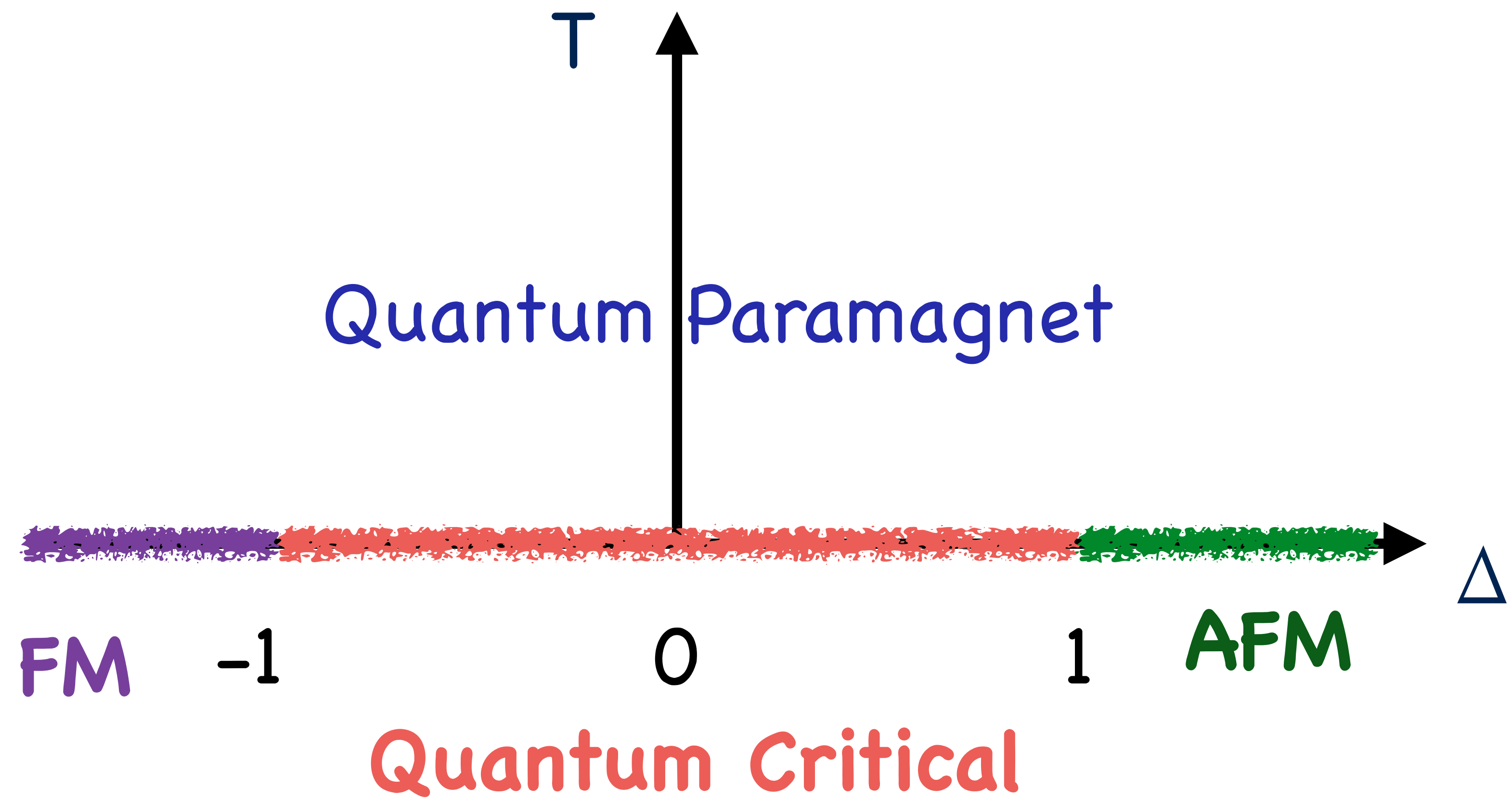
Cases with $\xi \to \infty$ and/or $\xi \gtrsim \ell$ will be most interesting.

Ground state of critical spin-1/2 XXZ chain

$$H = J \sum_{j=1}^{L} S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta\, S_j^z S_{j+1}^z$$

$-1 < \Delta \leq 1$

$$\Delta = -\cos(\pi\eta)$$



T=0: Quasi-long-range **AFM** order in the xy-plane for -1<Δ≤1:

$$\Delta = -\cos(\pi\eta)$$

$$\langle \mathrm{GS}|S^x_{j+n}S^x_j|\mathrm{GS}\rangle = (-1)^n \frac{A}{4n^\eta}\left(1 - \frac{B}{n^{4/\eta-4}}\right) - \frac{\tilde{A}}{4n^{\eta+1/\eta}}\left(1 + \frac{\tilde{B}}{n^{2/\eta-2}}\right) + \dots \,,$$

$$\langle \mathrm{GS}|S^z_{j+n}S^z_j|\mathrm{GS}\rangle = -\frac{1}{4\pi^2\eta n^2}\left(1 + \frac{\tilde{B}_z}{n^{4/\eta-4}}\frac{4-3\eta}{2-2\eta}\right) + (-1)^n \frac{A_z}{4n^{1/\eta}}\left(1 - \frac{B_z}{n^{2/\eta-2}}\right) + \dots$$

for |n|≫1

N.B. Slowest decay close to **ferro**magnet Δ≈-1 !

Subsystem Magnetization

smooth

$$S^\alpha(\ell) = \sum_{j=1}^{\ell} S_j^\alpha \ ,$$

staggered

$$N^\alpha(\ell) = \sum_{j=1}^{\ell} (-1)^j S_j^\alpha$$

Probability distributions:

$$P_N^\alpha(m,\ell) = \langle \mathrm{GS}|\delta\big(N^\alpha(\ell) - m\big)|\mathrm{GS}\rangle = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{-i\theta m} \underbrace{\langle \mathrm{GS}|e^{i\theta N^\alpha(\ell)}|\mathrm{GS}\rangle}_{F_\ell^\alpha(\theta)}$$

Generating function

$$P_N^\alpha(m,\ell) = \begin{cases} \sum_{r\in\mathbb{Z}} \widetilde{F}_\ell^\alpha(r)\ \delta(m-r) & \text{if } \ell \text{ is even,} \\ \sum_{r\in\mathbb{Z}} \widetilde{F}_\ell^\alpha(r+\frac{1}{2})\ \delta\big(m-r-\frac{1}{2}\big) & \text{if } \ell \text{ is odd.} \end{cases}$$

Universality

$$F_\ell^\alpha(\theta) = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \langle \mathrm{GS}|\big(N^\alpha(\ell)\big)^n|\mathrm{GS}\rangle$$

moments are not universal
(easy to see by bosonization)

But **ratios** $\dfrac{\left\langle \, \big(N^\alpha(\ell)\big)^{2n} \, \right\rangle}{\left\langle \, \big(N^\alpha(\ell)\big)^{2} \, \right\rangle^n}$ are!

"Rescaled" generating function is universal:

$$\langle \mathrm{GS}|e^{i\theta' N^\alpha(\ell)}|\mathrm{GS}\rangle \; , \qquad \theta' = \frac{\theta}{\sqrt{\langle \mathrm{GS}|\big(N^\alpha(\ell)\big)^2|\mathrm{GS}\rangle}}$$

Have studied $F_\ell(\theta)$ and $G_\ell(\theta)$ by a combination of numerical (iTEBD) and analytic (exact field theory, free fermion, 2-loop RG) methods.

Field theory approach based on

Low-energy field theory

$$\mathcal{H}(\Delta) = \frac{v}{2}\int dx \left[K(\partial_x\theta)^2 + \frac{1}{K}(\partial_x\phi)^2\right]$$

Lattice spin operators

$$S_j^z \simeq -\frac{a_0}{\sqrt{\pi}}\partial_x\phi(x) + (-1)^j c_1 \sin(\sqrt{4\pi}\phi(x)) + \ldots,$$

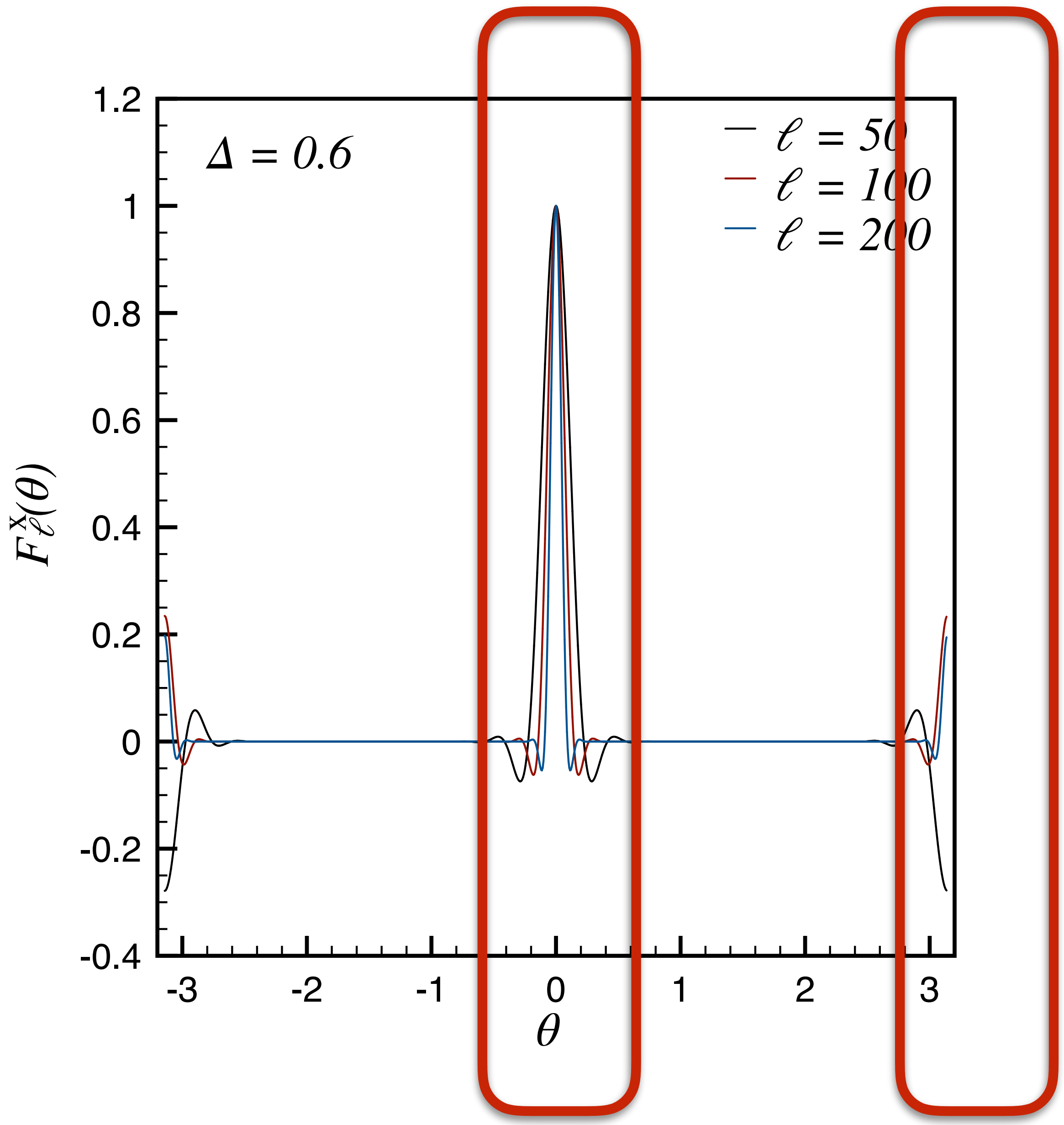
$$S_j^x \simeq b_0(-1)^j \cos\left(\sqrt{\pi}\,\theta(x)\right) + ib_1 \sin\left(\sqrt{\pi}\,\theta(x)\right)\sin\left(\sqrt{4\pi}\phi(x)\right) + \ldots$$

Characteristic function

$$F_\ell^x(\theta) \approx \langle 0|e^{-i\theta\frac{b_0}{a_0}\int_0^\ell dx\ \cos\left(\sqrt{\pi}\phi(x)\right)}|0\rangle$$

Results for Staggered Subsystem Magnetizations

A. Transverse staggered magnetization (“order parameter”)

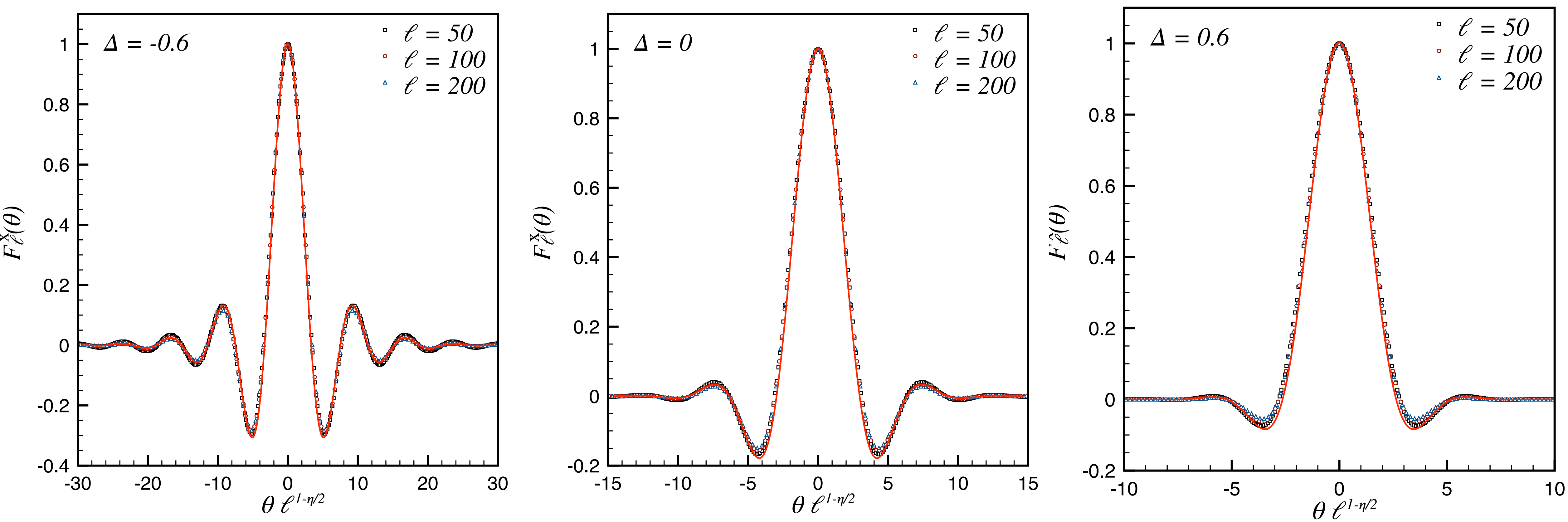


very small
except at θ≈0,π

Observe **scaling collapse** around θ=0,π:

Scaling collapse around θ≈0:

$$F_\ell^x(\theta) = F^x(\theta \ell^{1-\eta/2})$$



Red lines: exact field theory results.

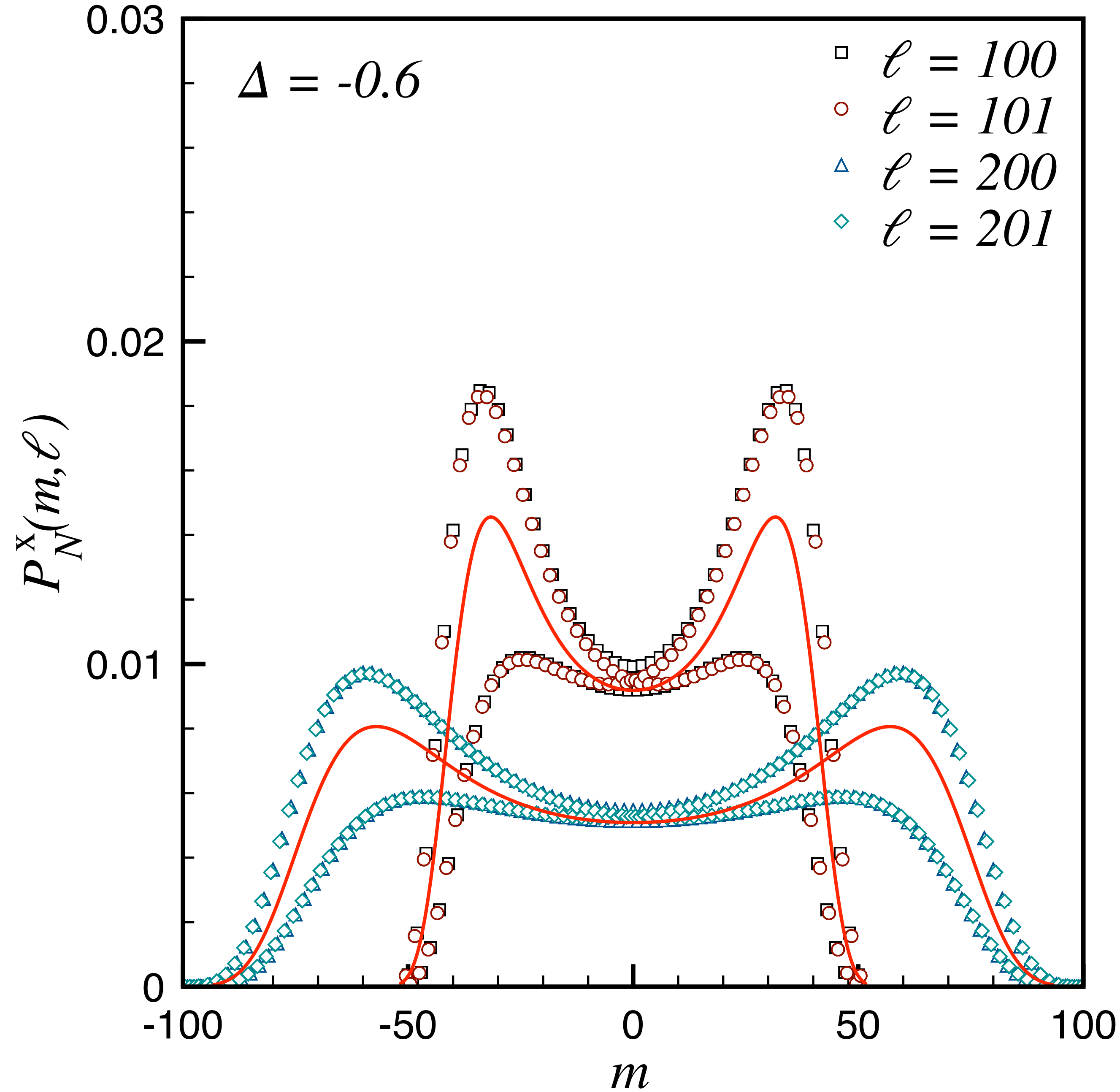
Similar scaling collapse around θ≈π (subleading contribution)

Probability distribution:

- small
- broad
- bimodal

- even/odd effect in m
- even/odd effect in ℓ

go away for large ℓ

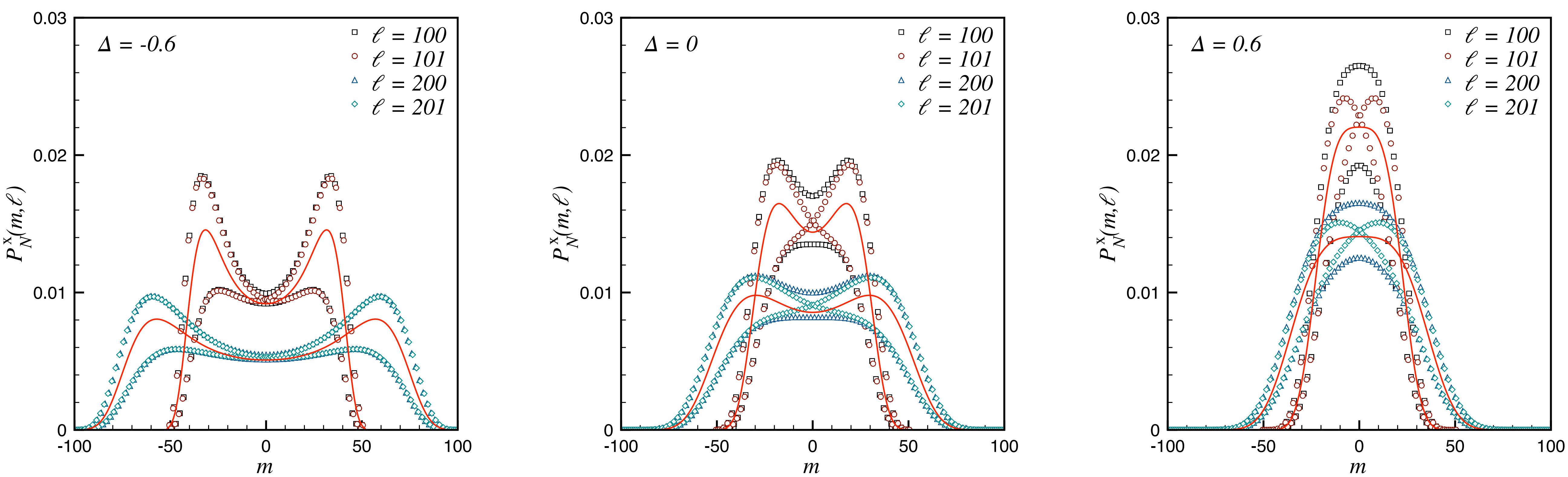


$$P_N^x(m,\ell) \simeq \ell^{\frac{\eta}{2}-1}\widetilde{\mathcal{F}}_0^x(m/\ell^{1-\eta/2}) + (-1)^{\lfloor \ell/2 \rfloor + \lfloor m \rfloor}\ell^{\frac{\eta}{2}-5/4}\widetilde{\mathcal{F}}_{e/o}^x(m/\ell^{1-\eta/2})$$

red lines, field theory results

subleading

Evolution along the critical line -1<Δ≤1:

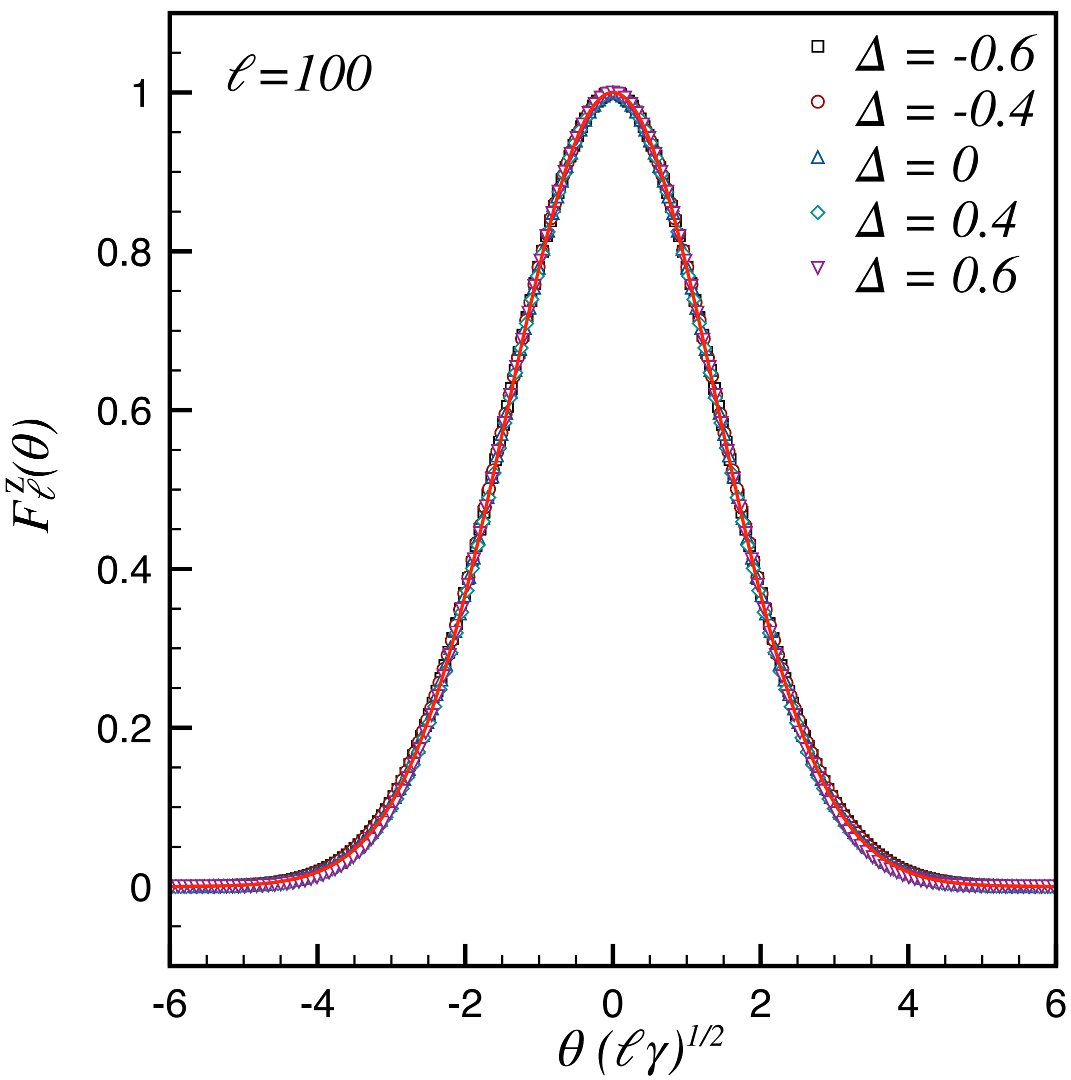


B. Longitudinal staggered subsystem magnetization

Scaling collapse around θ≈0:

$$F_\ell^z(\theta \approx 0) = e^{-\gamma z^2/4}\ , \quad z = \theta \ell^{1/2}$$

simple Gaussian



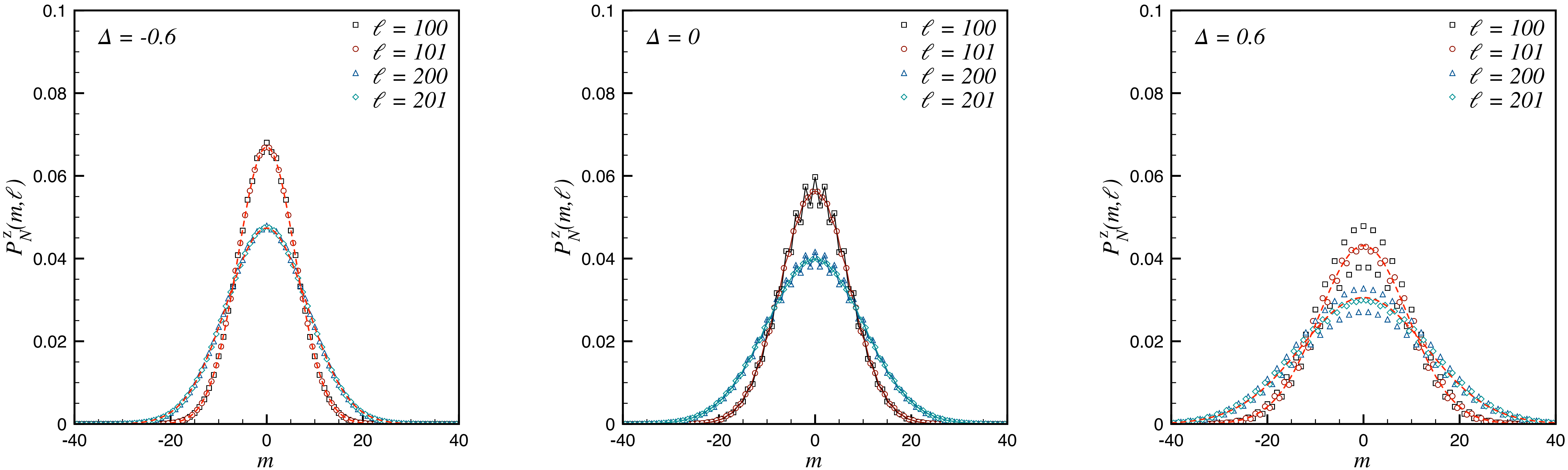
find that rescaling parameter is

$$\gamma = \frac{1}{2 - 2\eta}$$

red line = exact results

Similar scaling collapse around θ≈π (subleading contribution)

Probability distribution:



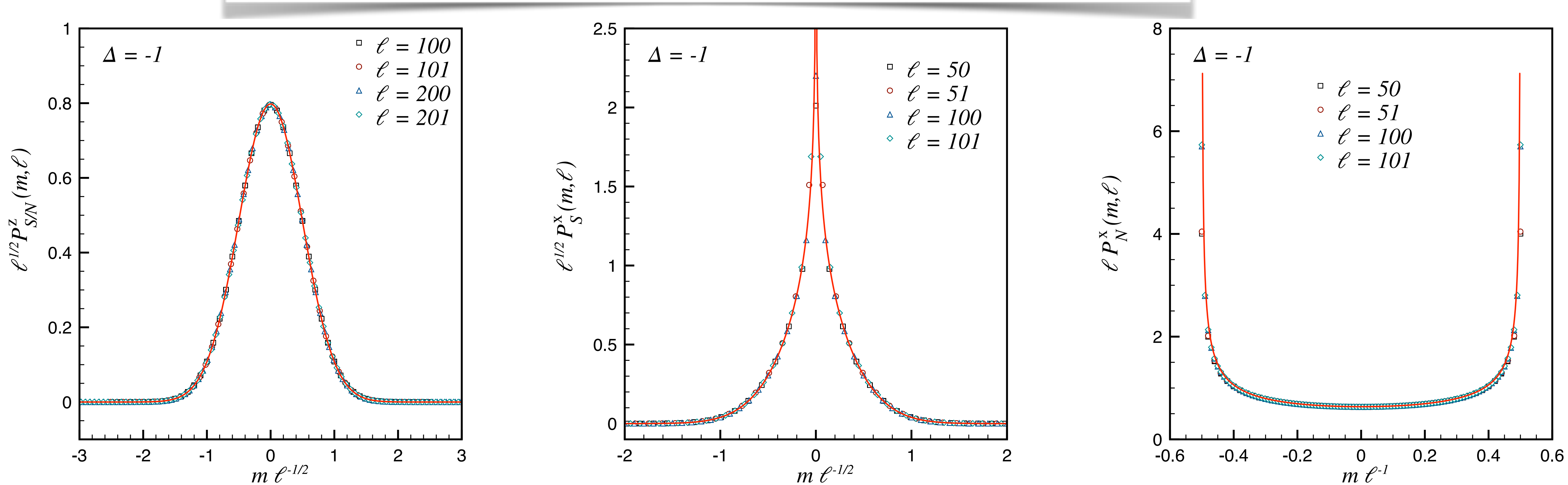
PD is Gaussian and much narrower than transverse analog.

The limit Δ→-1

Ground state = equal amplitude superposition of all $S^z=0$ states
→ closed form expressions for generating functions

$$P_S^z(m,\ell) = P_N^z(m,\ell) = \ell^{-1/2}\sqrt{2/\pi}\, \mathrm{e}^{-2m^2/\ell},$$
$$P_S^x(m,\ell) = \ell^{-1/2}\sqrt{2/\pi^3}\, \mathrm{e}^{-m^2/\ell} K_0(m^2/\ell),$$
$$P_N^x(m,\ell) = \ell^{-1}\frac{2}{\pi\sqrt{1-4m^2/\ell^2}},$$

for large ℓ



Melting of LRO after a Quantum Quench

$$H = J \sum_{j=1}^{L} S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta\, S_j^z S_{j+1}^z$$

Initial state: $|\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\ldots\rangle$

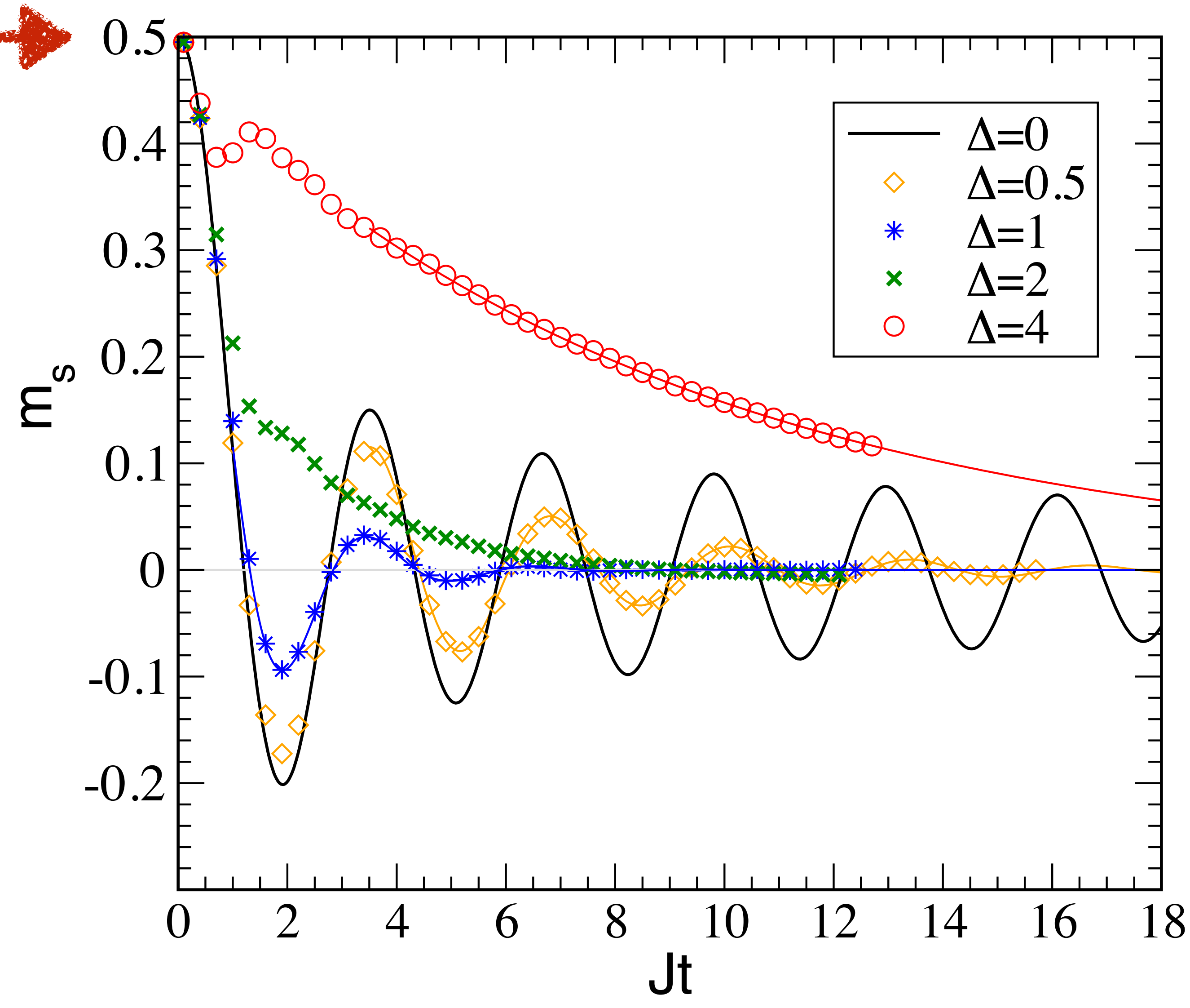
AFM Long-range order

Time-evolve with H, measure prob. dist. of $N_\ell^z = \sum_{j=1}^{\ell} (-1)^j S_j^z$

expectation value of N_ℓ^z

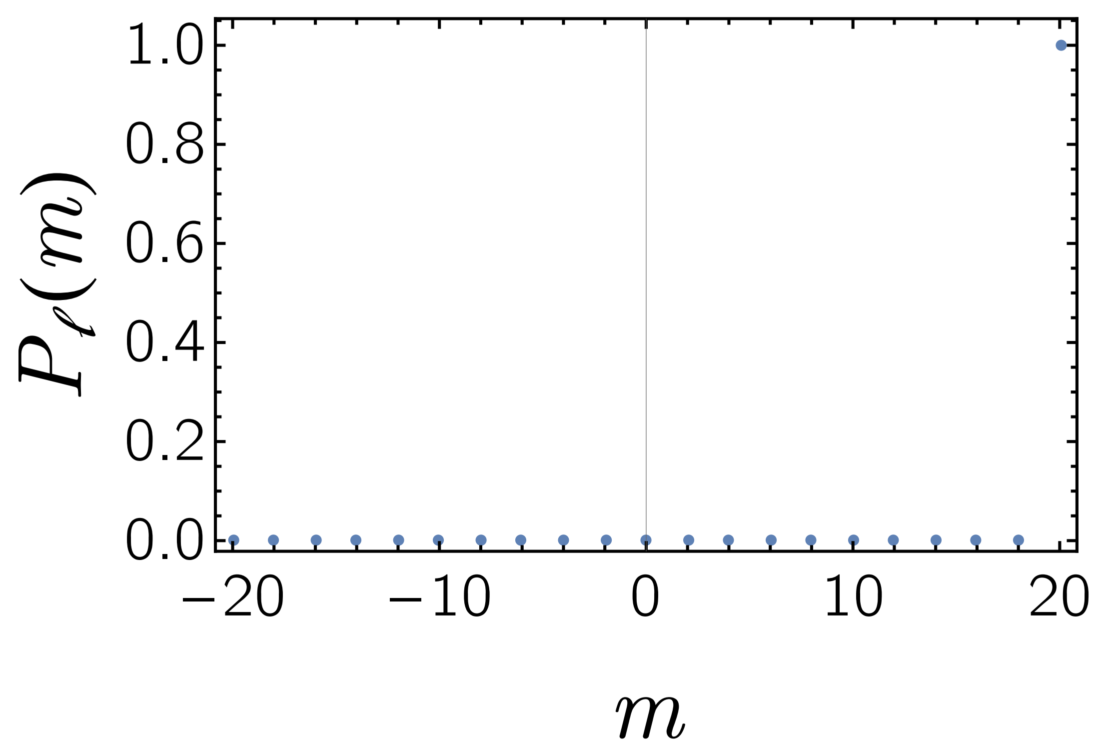
Barmettler et al '09

LRO



LRO gone

Probability distribution in initial state:



delta-function at m=ℓ/2

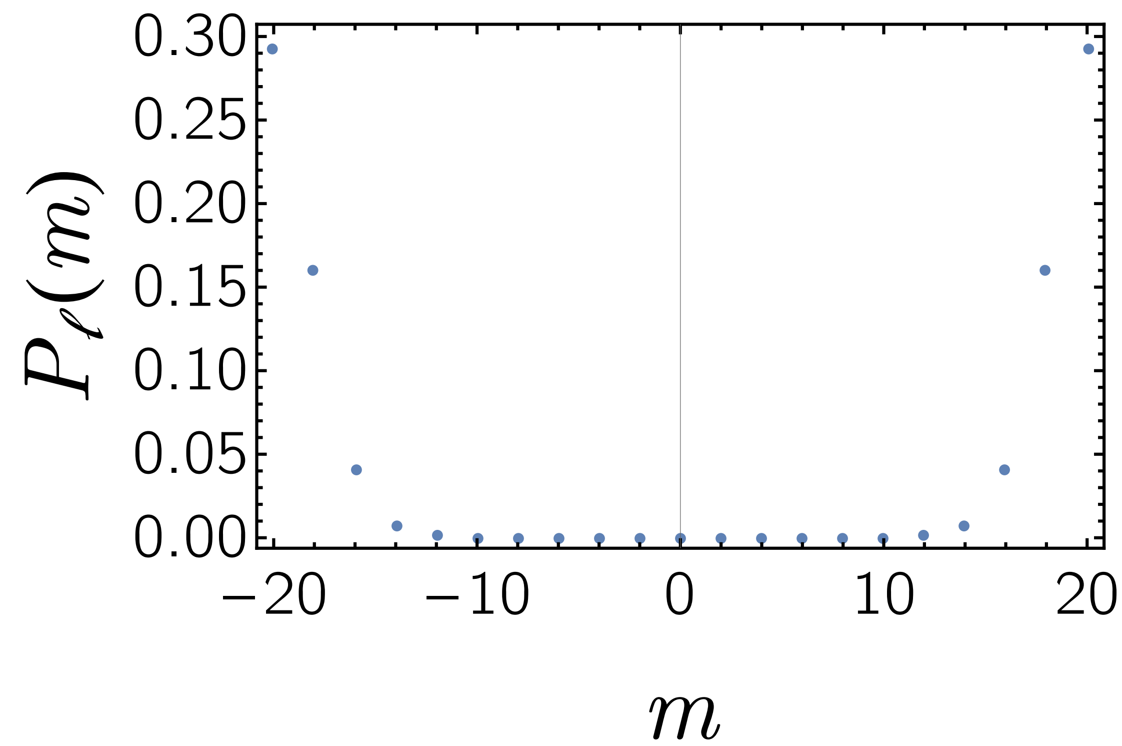
What do we expect in the stationary state?

Stationary State:

Finite correlation length ξ

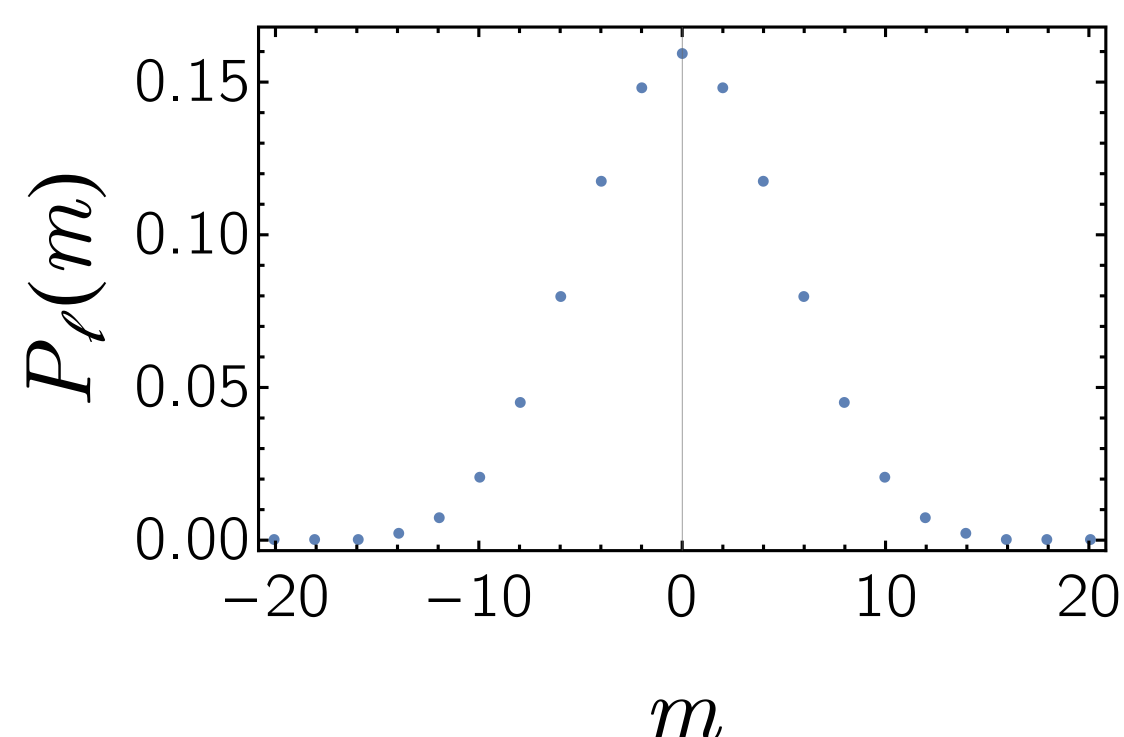
$\xi > \ell$

"small quench"

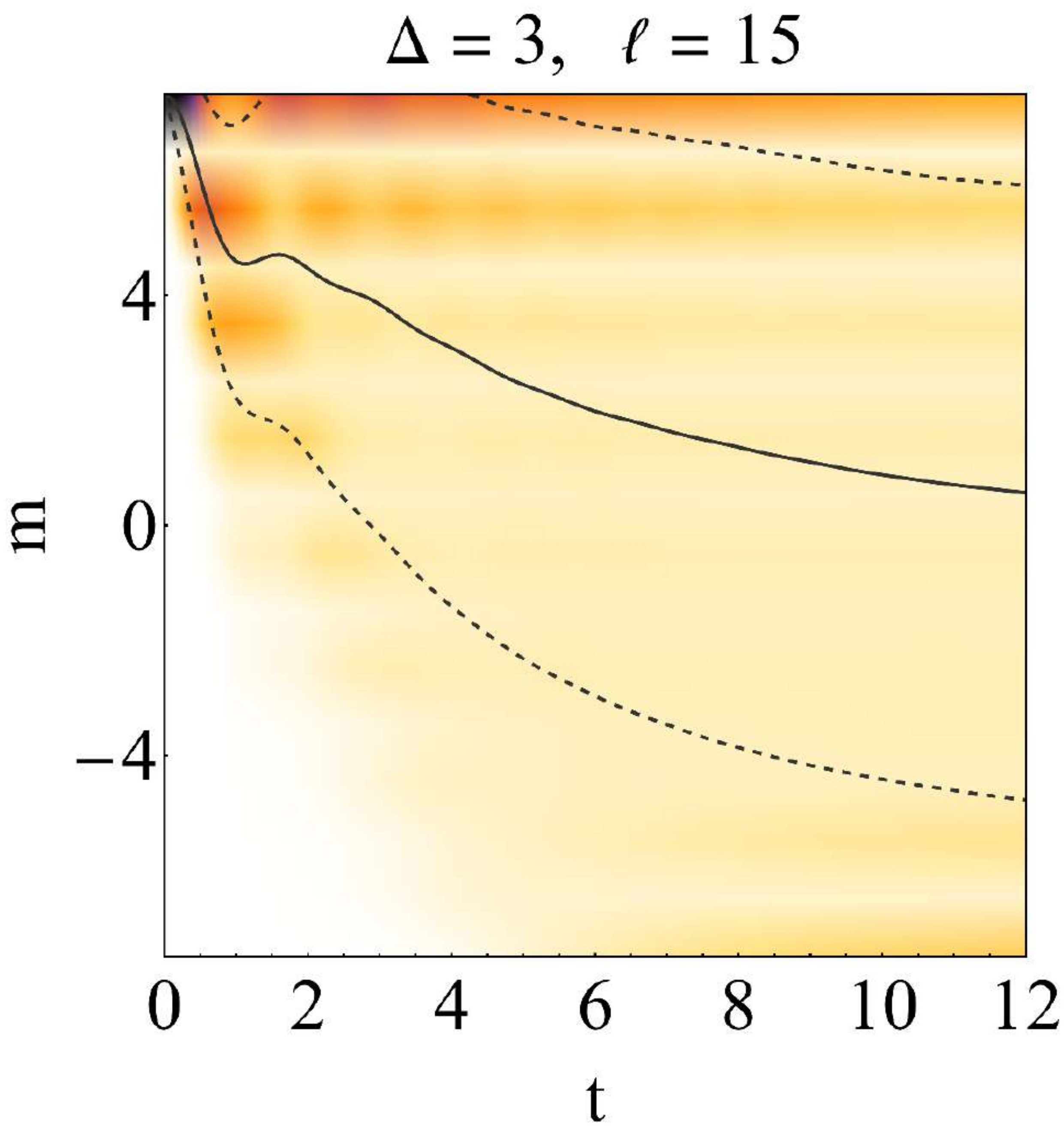


$\xi < \ell$

"large quench"



Time evolution for a "small quench"



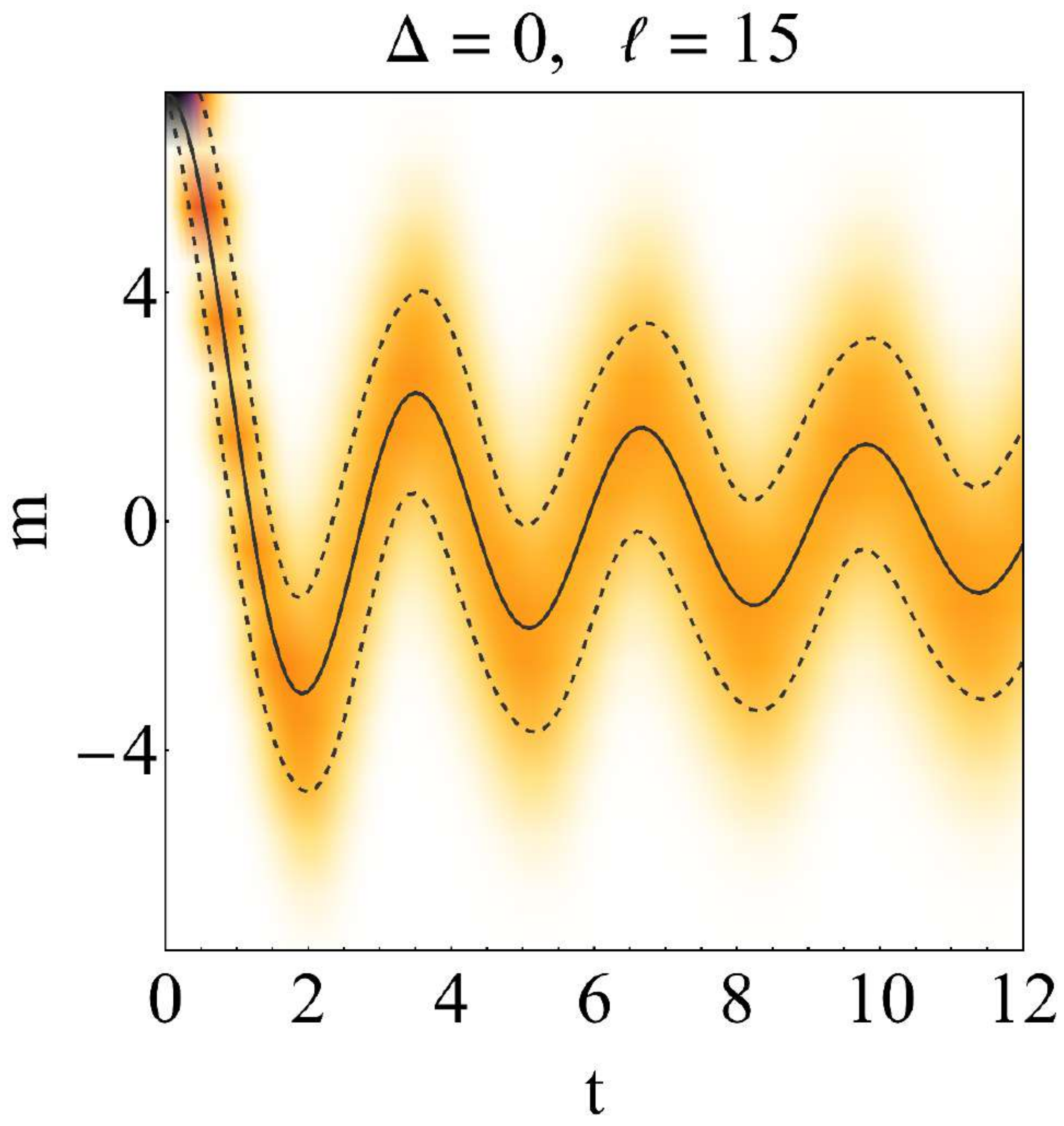
peak at m=ℓ/2 remains

average

peak at m=−ℓ/2 develops

Short-range order remains in the stationary state

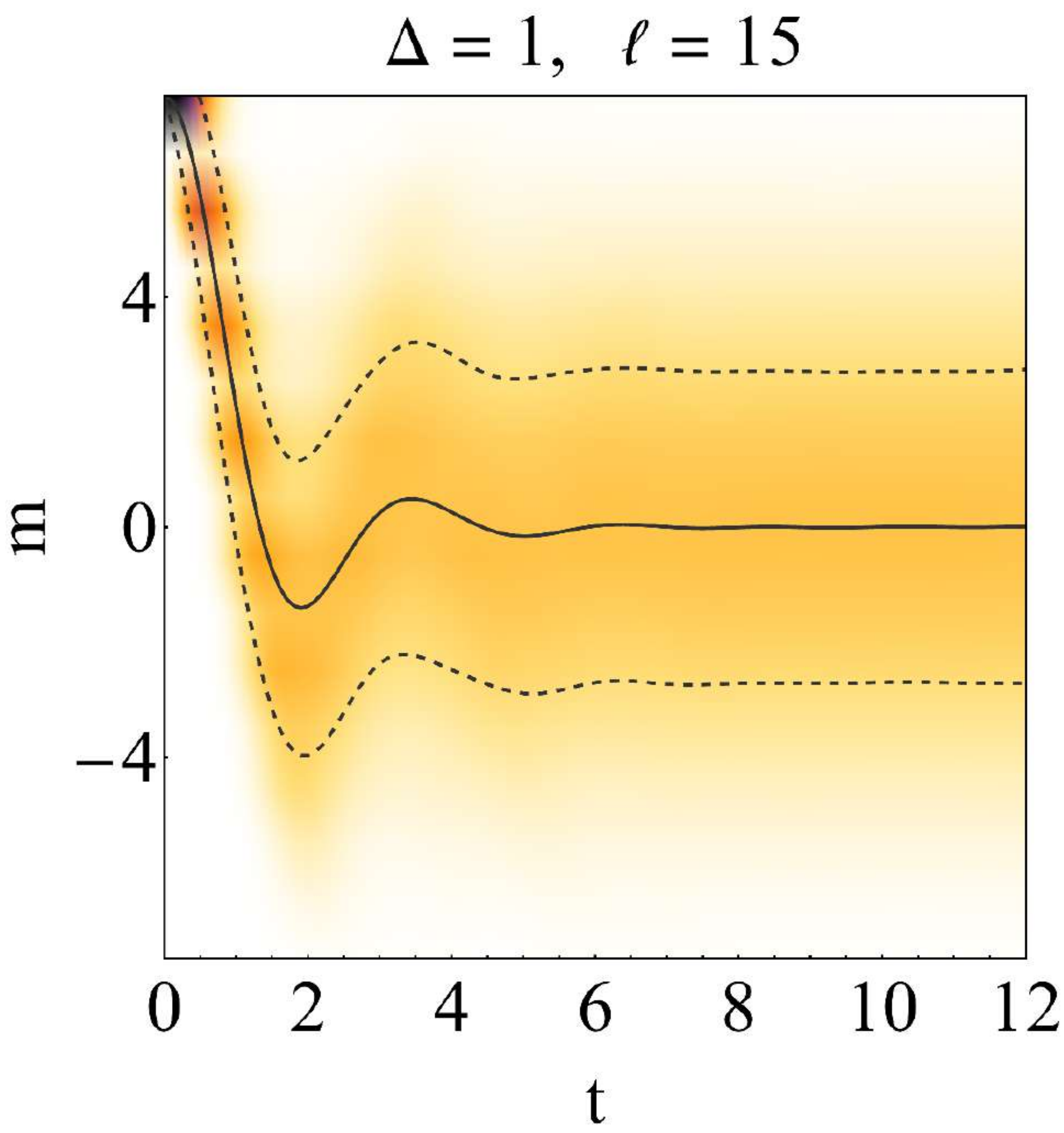
Time evolution for a "large quench"



Prob. dist. = narrow Gaussian

average

Time evolution for an "intermediate quench"



Broad prob. dist.

average

Analytic results for large Δ

$$\lim_{t\to\infty} P_\ell(m,t)$$

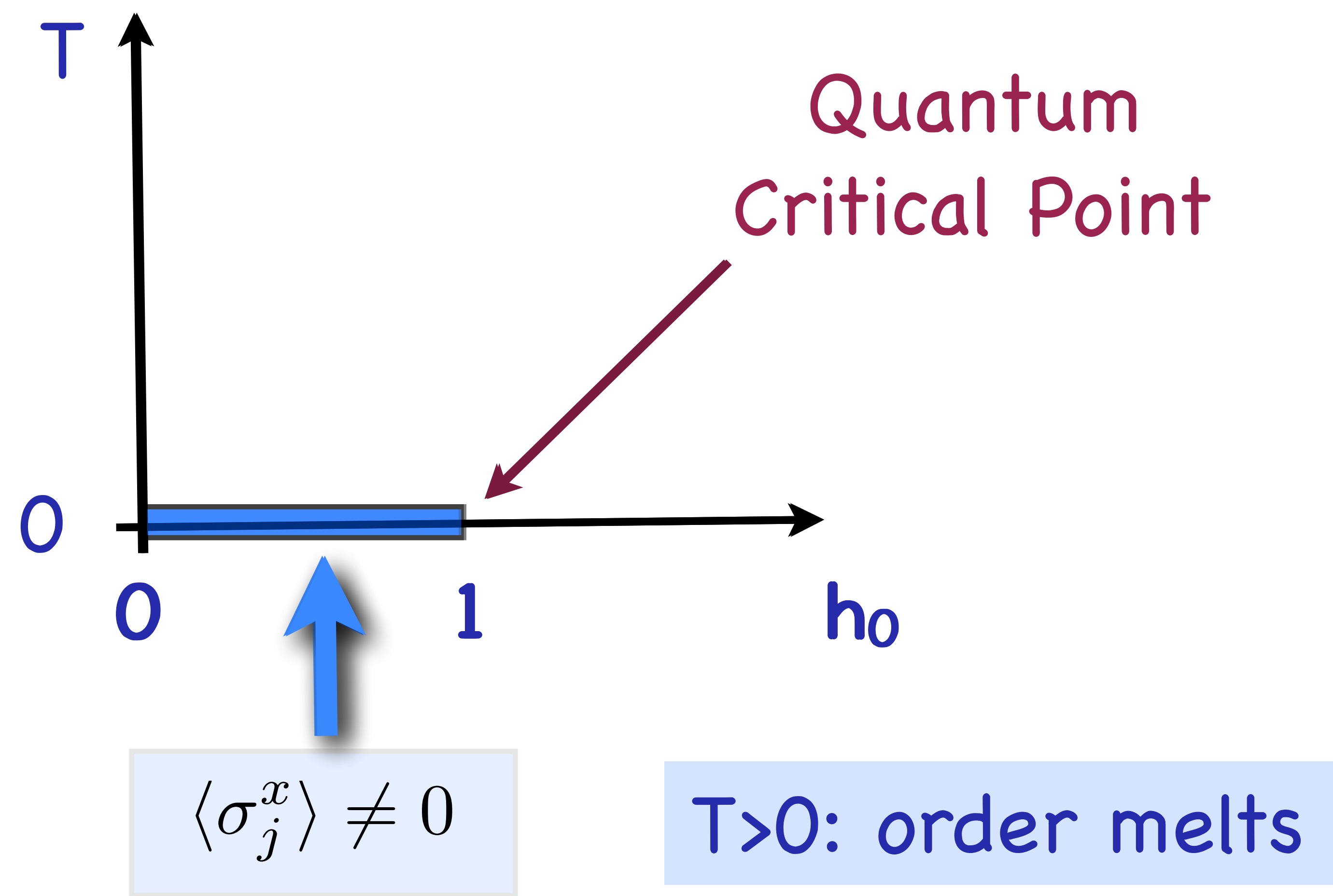
by combining low-density expansion for truncated GGE with $1/\Delta$-expansion

Analytic results: Transverse-Field Ising Chain

$$H(h) = -\sum_{j=-\infty}^{\infty} \left[\sigma_j^x \sigma_{j+1}^x + h\sigma_j^z\right].$$

$\mathbb{Z}_2$ symmetry: rotations around z-axis by π $\quad \sigma_j^x \to -\sigma_j^x$

Phase Diagram:



Probability distribution of **transverse** magnetisation

$$S^z_u(\ell) = \sum_{j=1}^{\ell} \sigma^z_j = \sum_{j=1}^{\ell} 1 - 2c^\dagger_j c_j$$

local in fermions

$$P^{(u)}(m) = \mathrm{Tr}[\rho\delta(m - S^z_u(\ell)] = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \mathrm{Tr}[\rho e^{i\lambda S^z_u(\ell)}]$$

ρ: density matrix

$$= 2\sum_{r\in\mathbb{Z}} P^{(u)}_w(r)\delta(m - 2r) \qquad (\ell \text{ even})$$

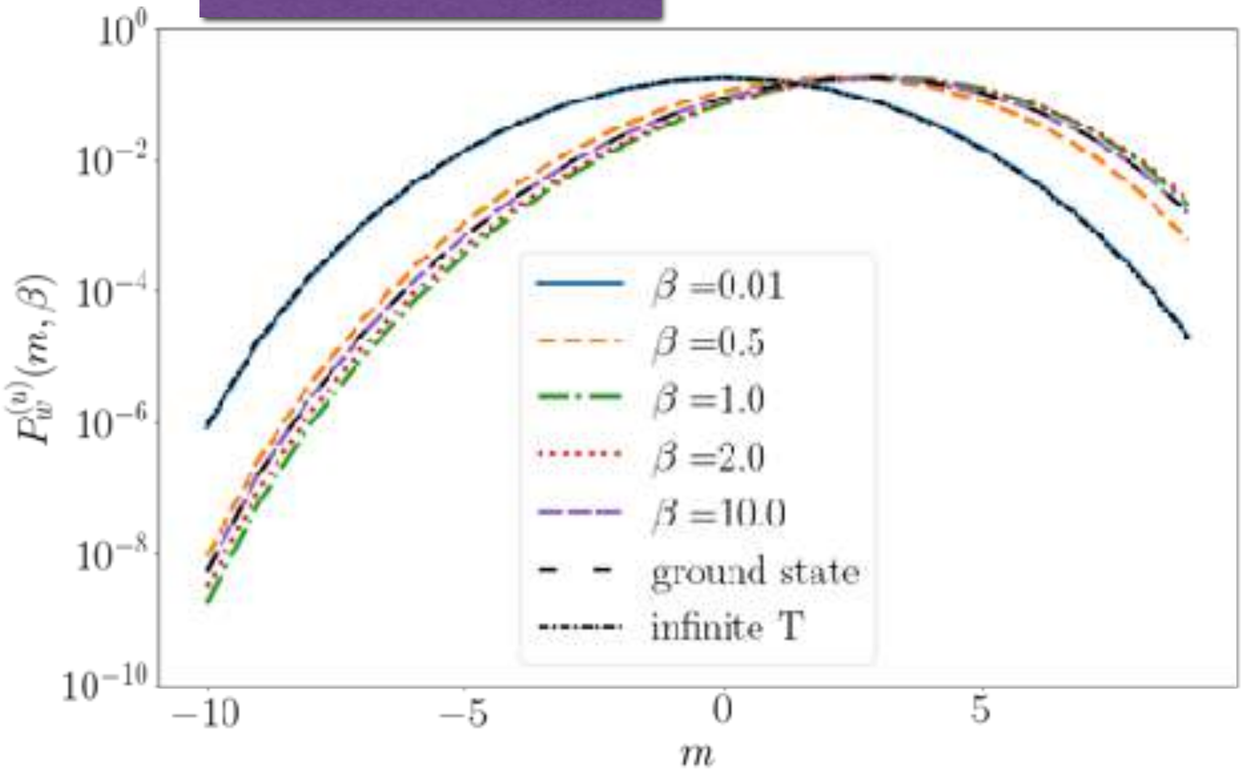
plot these

Calculated this for **equilibrium states**

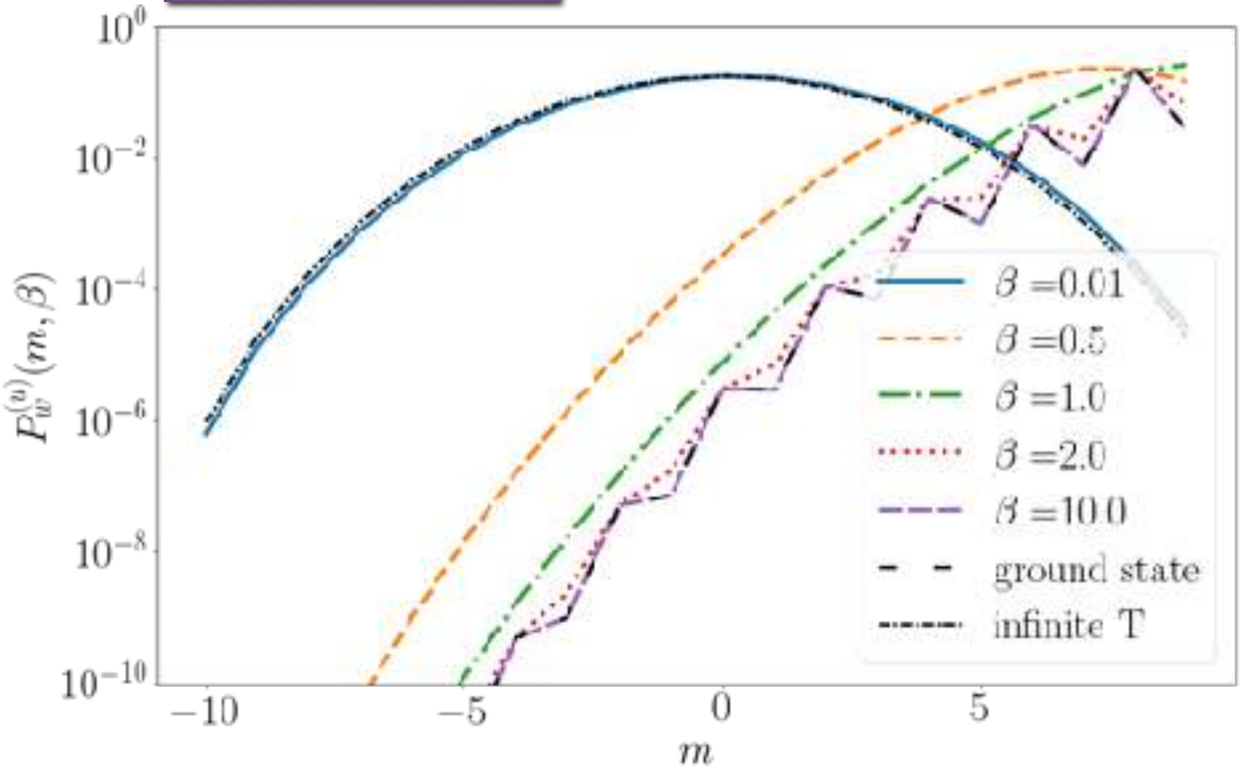
& after **quantum quenches** [ρ→ρ(t)]

Finite temperature equilibrium states

h=0.5, ℓ=20



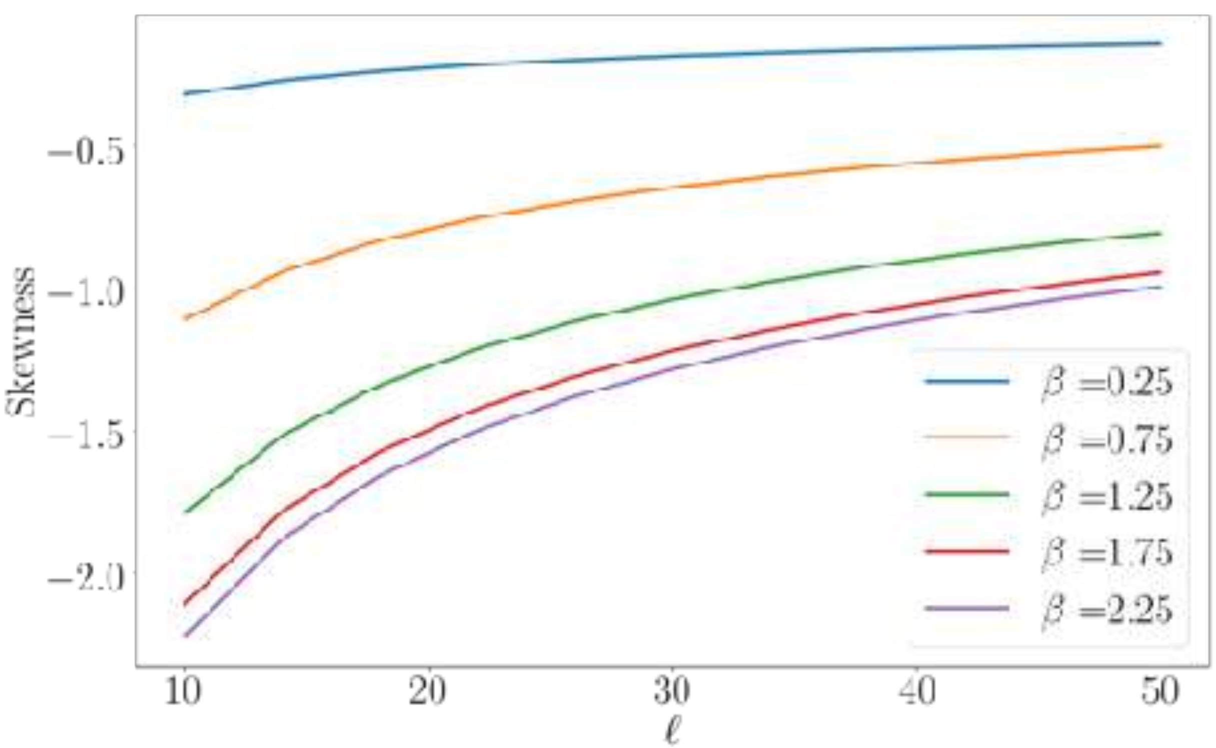
h=2, ℓ=20

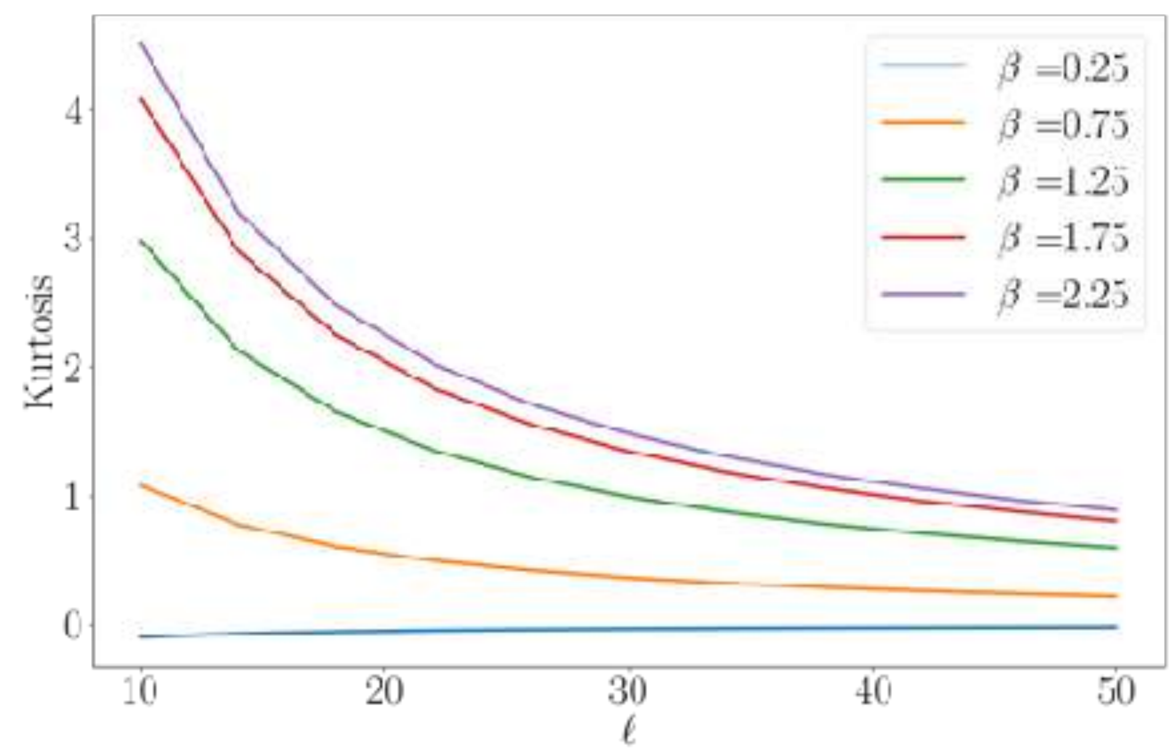


Generally non-Gaussian

Skewness & Excess Kurtosis for h=2 and several temperatures

$$\left\langle \left[\frac{X}{\sqrt{\langle X^2 \rangle_\beta}} \right]^3 \right\rangle_\beta , \quad \left\langle \left[\frac{X}{\sqrt{\langle X^2 \rangle_\beta}} \right]^4 \right\rangle_\beta - 3 , \quad X = S_u^z(\ell) - \langle S_u^z(\ell) \rangle_\beta$$

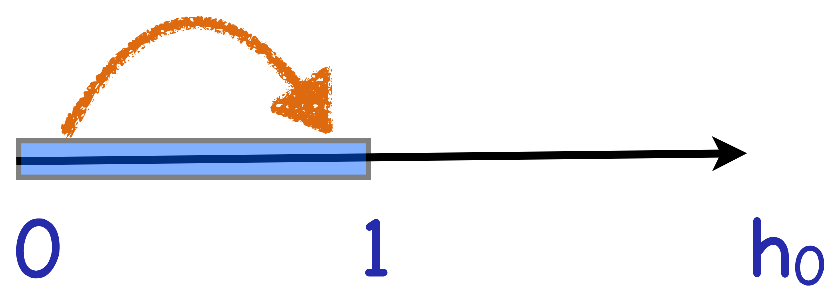


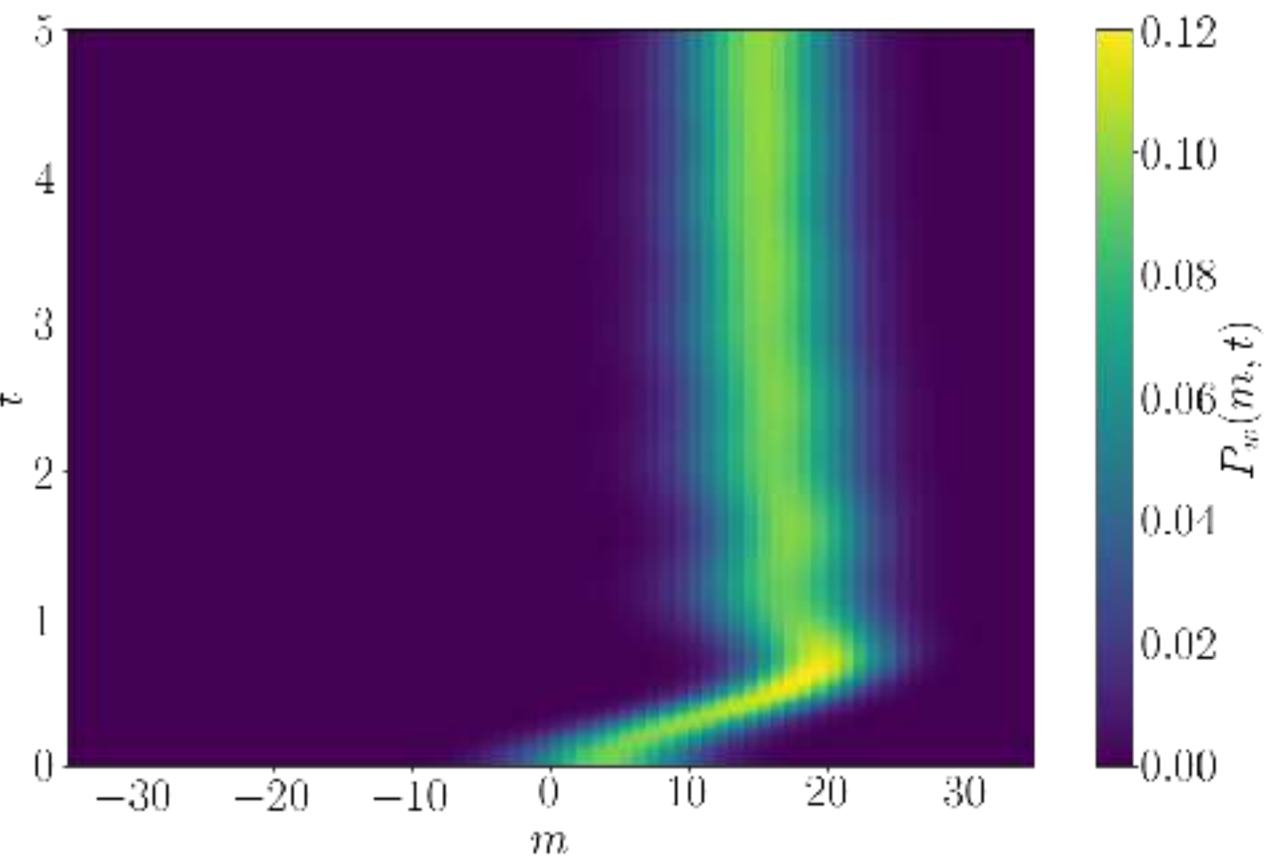


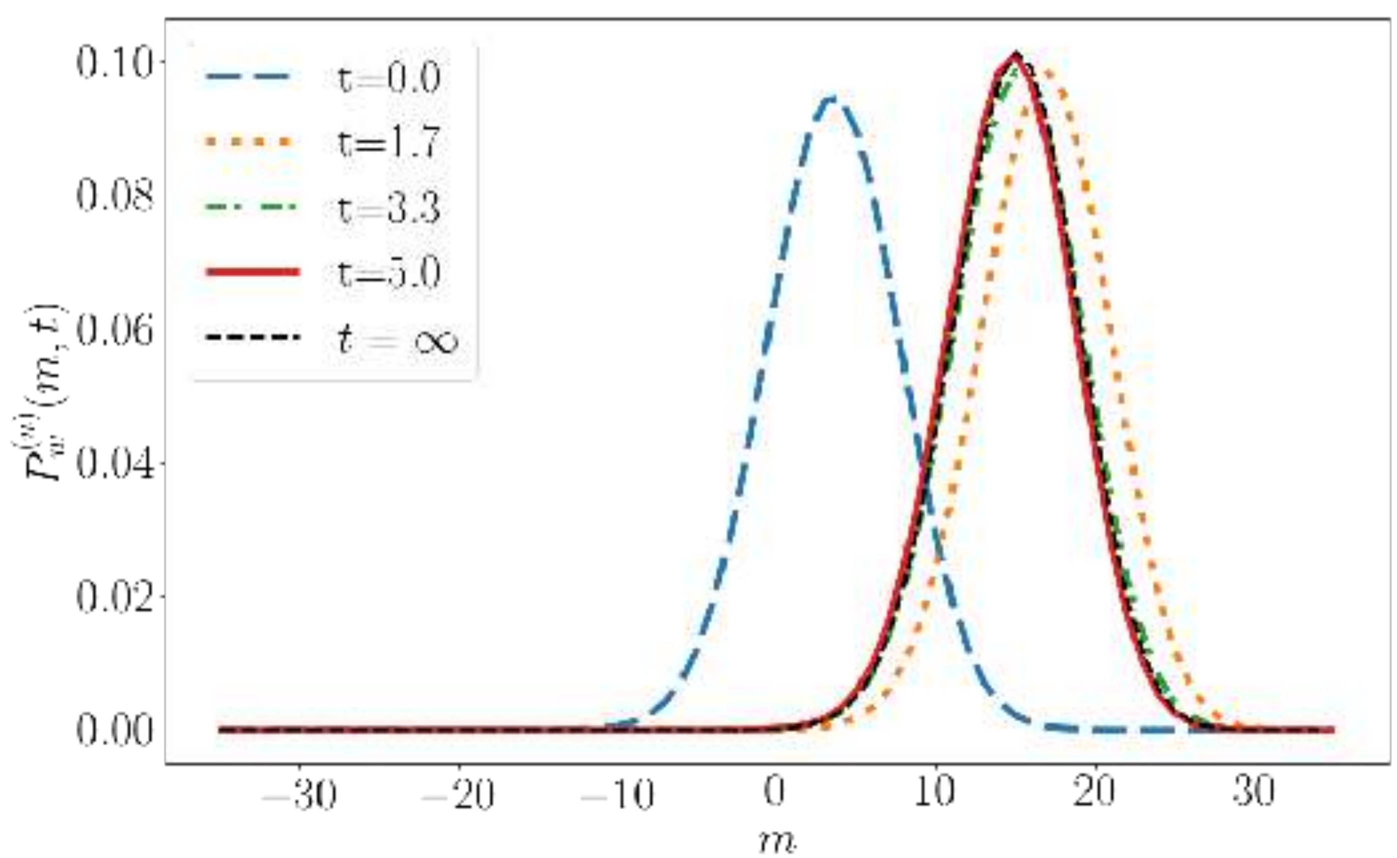
Prob. dist. becomes more Gaussian as we increase ℓ or T ✔

Transverse field quench: prepare system in ground state of $H(h_0)$, time evolve with $H(h)$

$h_0=0.2$, $h=0.8$

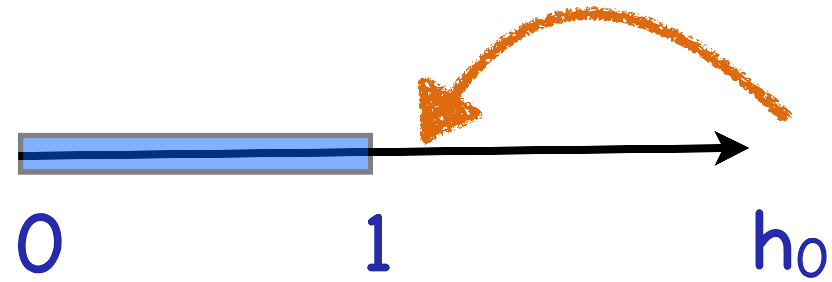


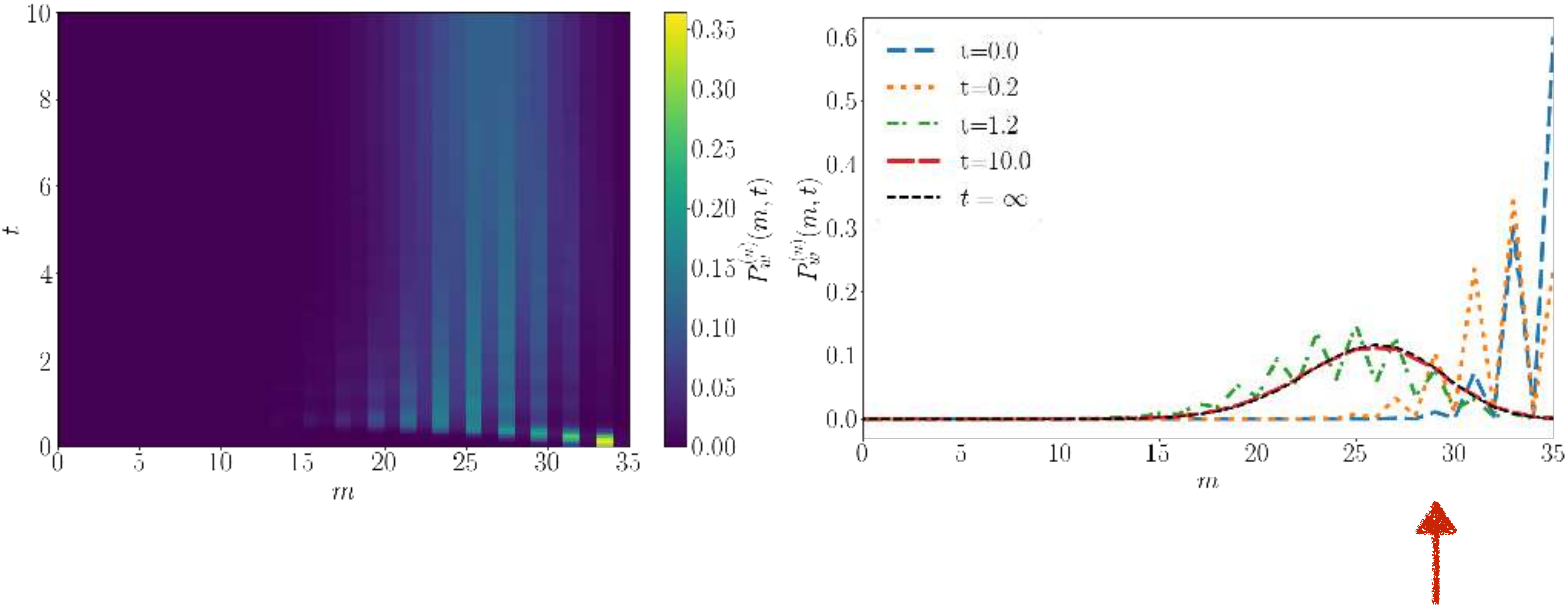




Covfefe.

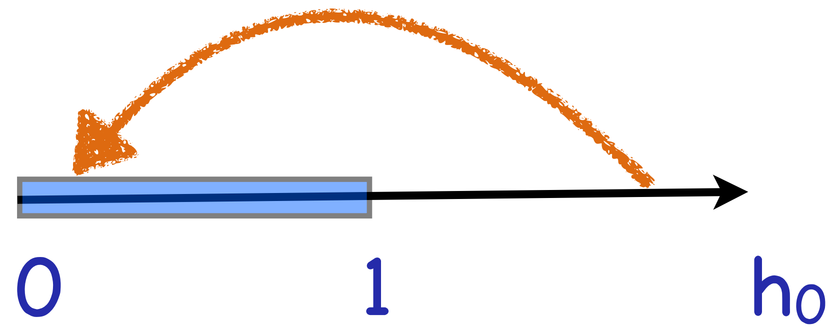
$h_0=3$, $h=1.2$

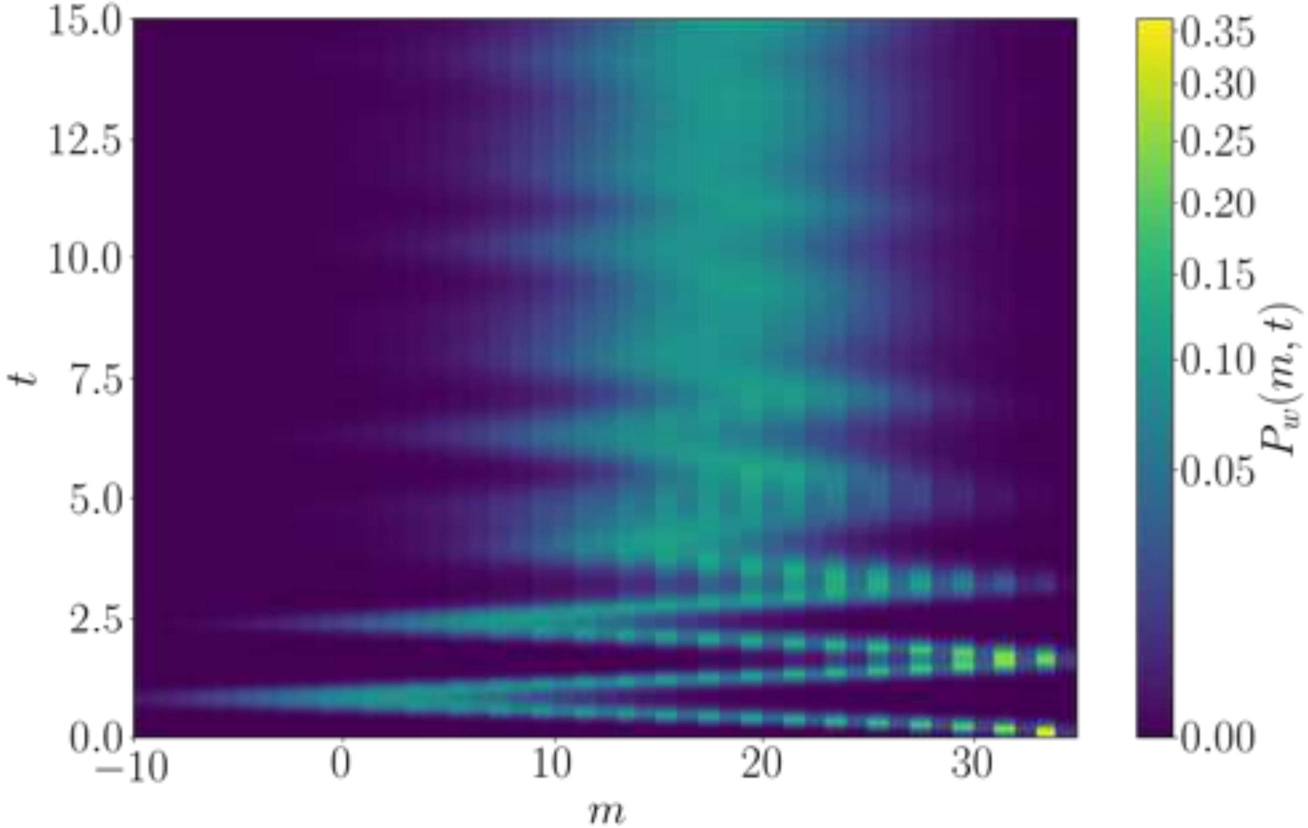


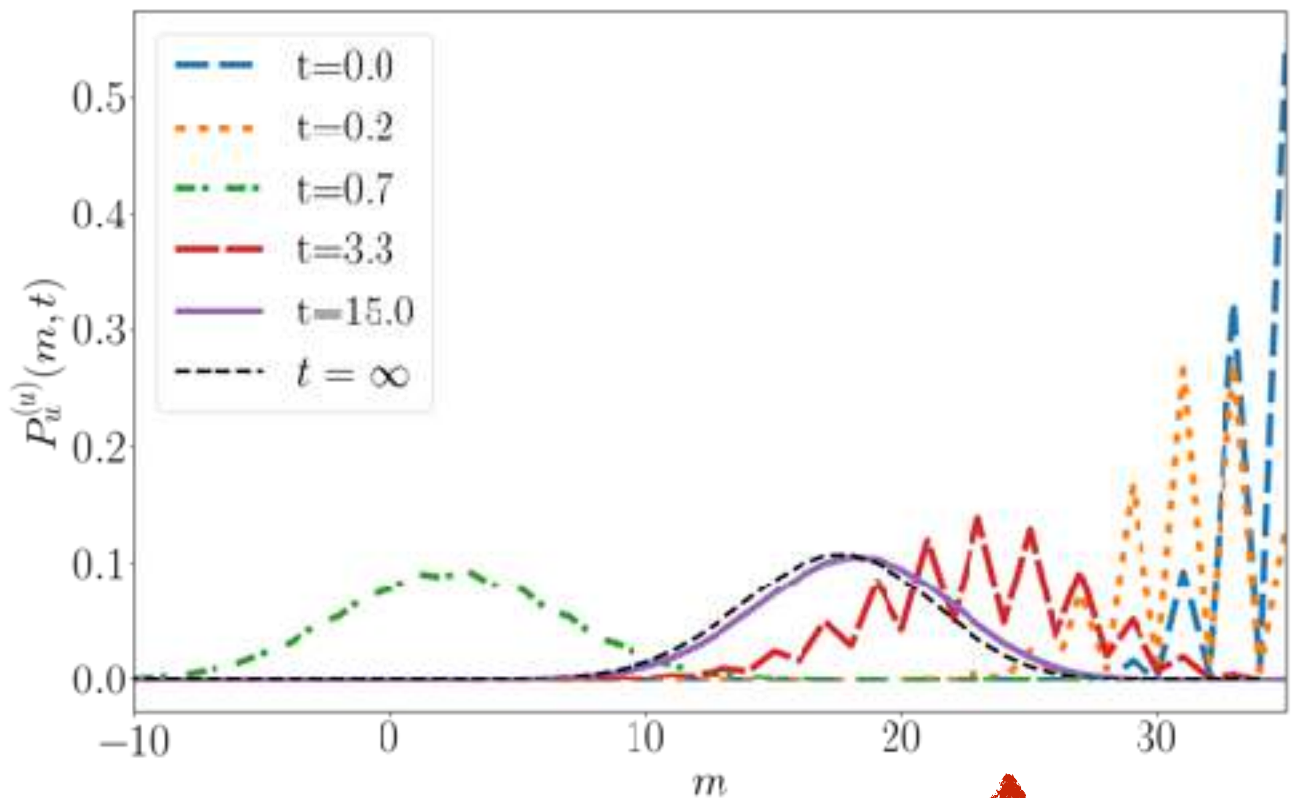


even/odd structure that washes out over time

$h_0=3,\ h=0.2$





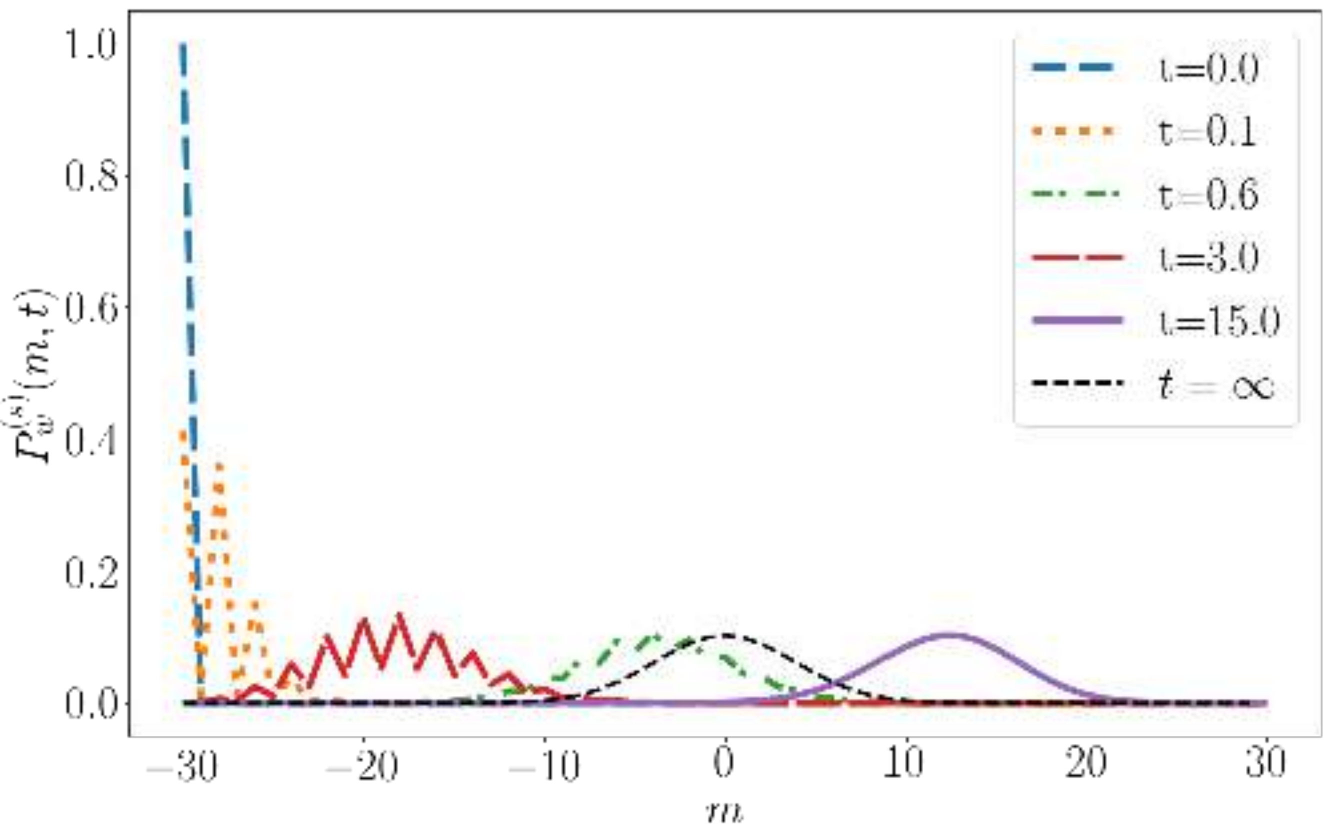


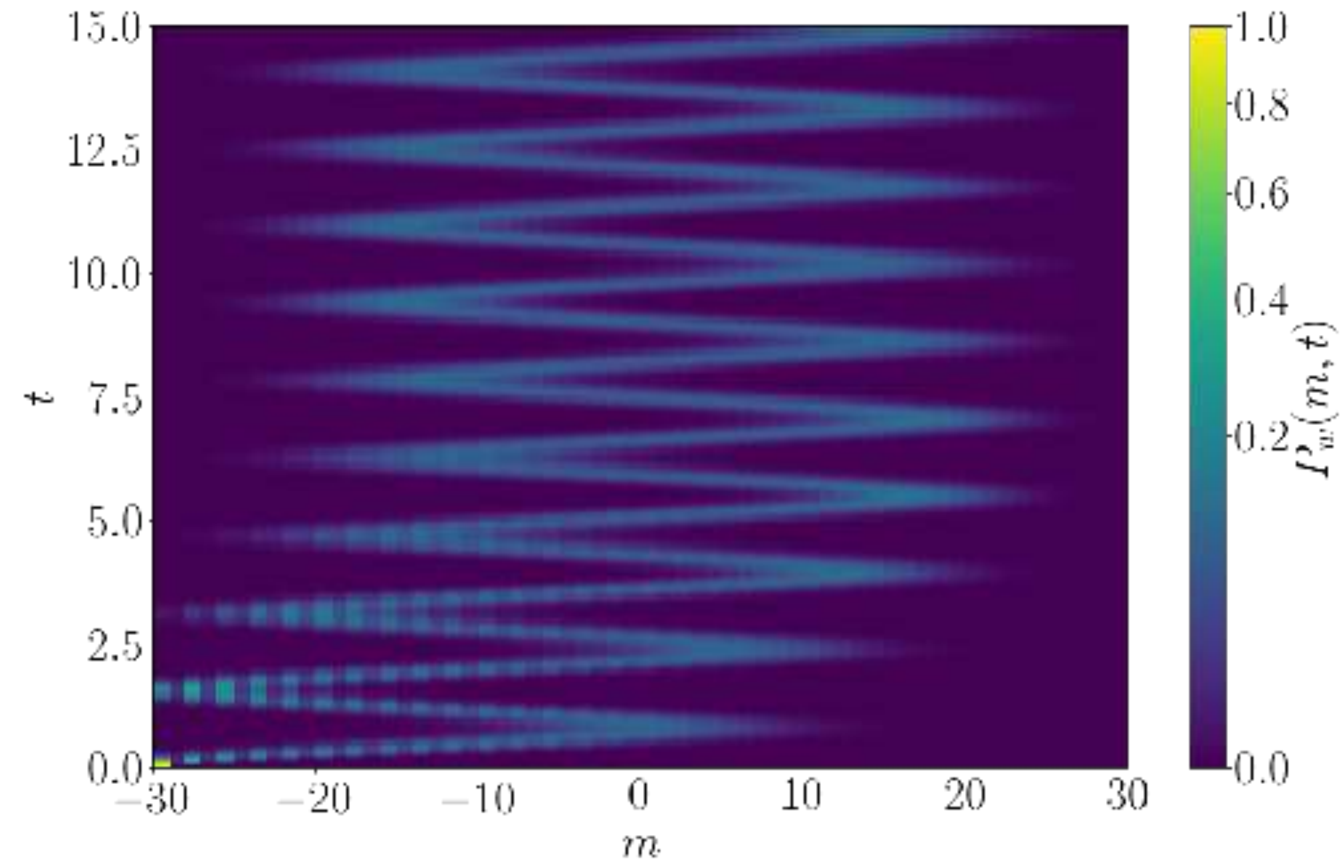
even/odd structure that washes out over time

Néel quench: prepare system in class. Néel state, time evolve with H(h), consider prob. dist. of staggered transverse magn.

$$S_s^z(\ell) = \sum_{j=1}^{\ell} (-1)^j \sigma_j^z$$

h=0.2





How to obtain analytic results: generalise

Fagotti&Calabrese '08
Calabrese, FHLE & Fagotti '12

Step 1: determinant representation for generating function

$$\chi^{(u)}(\lambda,\ell) = (2\cos\lambda)^{\ell}\sqrt{\det\left(\frac{1-\tan(\lambda)\Gamma'}{2}\right)},$$

← known 2ℓx2ℓ matrix

Step 2: multiple integral representation

$$\ln\chi^{(u)}(\lambda,\ell,t) = \ell\ln(\cos\lambda) - \frac{1}{2}\sum_{n=1}^{\infty}\frac{\left(\tan(\lambda)\right)^n}{n}\,\mathrm{Tr}\big[(\Gamma')^n\big]$$

$$\mathrm{Tr}\big[(\Gamma')^n\big] = \left(\frac{\ell}{2}\right)^n\int_{-\pi}^{\pi}\frac{dk_1\dots dk_n}{(2\pi)^n}\int_{-1}^{1}d\zeta_1\dots d\zeta_{n-1}\ \mu(\vec{\zeta})\ C(\vec{k})\ F(\vec{k})\ \exp\left(-i\ell\sum_{j=1}^{n-1}\frac{\zeta_j}{2}(k_j-k_0)\right)$$

Step 3: asymptotics from multi-dim stationary phase approx and summing result over all n

difficult.

Result:

$$\ln\chi(\lambda,\ell,t)\approx \ell\log(\cos\lambda)+\frac{\ell}{2}\sum_{n=0}^{\infty}\int_0^{2\pi}\frac{dk_0}{2\pi}\Theta(\ell-2n|v_k|t)\left[1-\frac{2n|v_k|t}{\ell}\right]\sum_{m=0}^{n+1}\cos\left(2m\varepsilon(k_0)t\right)f_{n,m}(\lambda,k_0)+\mathcal{C}$$

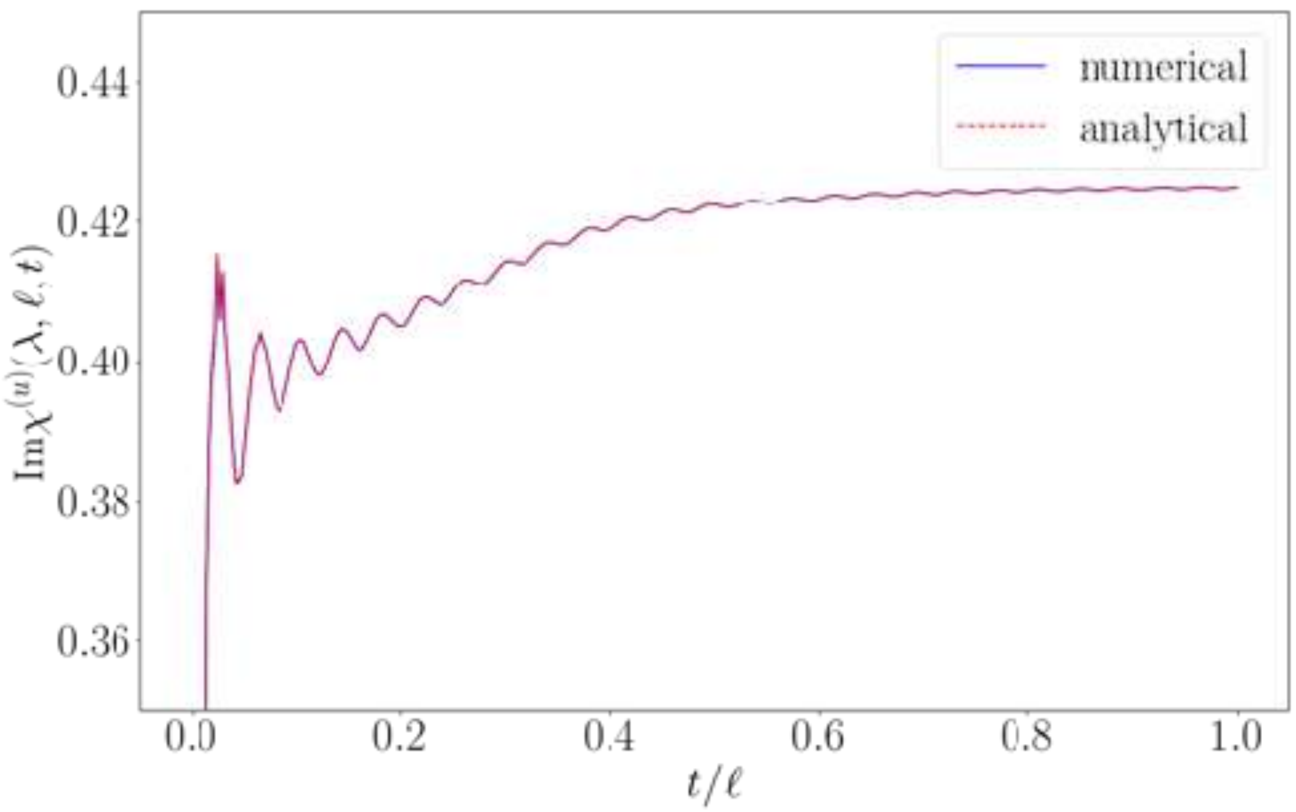
$$\begin{aligned}
f_{0,0}(\lambda,k_0) &= 2\ln\left(1+i\cos\Delta_{k_0}\tan\lambda e^{i\theta_{k_0}}\right)\,,\\
f_{1,0}(\lambda,k_0) &= \ln\left[1-\frac{\sin^2\Delta_{k_0}\tan^2\lambda(\cos\theta_{k_0}+i\cos\Delta_{k_0}\tan\lambda)^2}{(\sin^2\theta_{k_0}+(\cos\theta_{k_0}+i\cos\Delta_{k_0}\tan\lambda)^2)^2}\right]\,,\\
f_{2,0}(\lambda,k_0) &= \ln\left[1+\frac{\sin^4\Delta_{k_0}\tan^4\lambda\sin^2\theta_{k_0}(\cos\theta_{k_0}+i\cos\Delta_{k_0}\tan\lambda)^2}{((\sin^2\theta_{k_0}+(\cos\theta_{k_0}+i\cos\Delta_{k_0}\tan\lambda)^2)^2-\sin^2\Delta_{k_0}\tan^2\lambda(\cos\theta_{k_0}+i\cos\Delta_{k_0}\tan\lambda)^2)^2}\right]
\end{aligned}$$

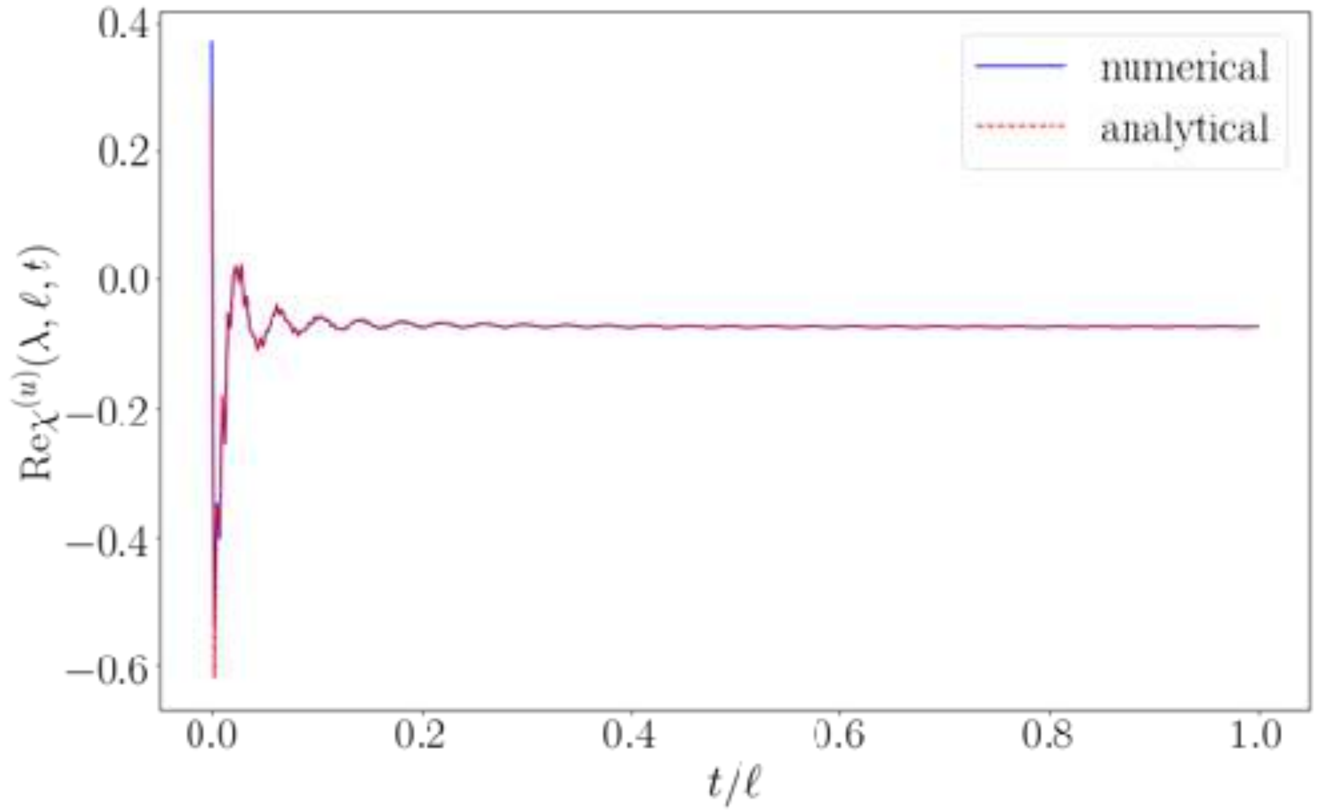
$$\begin{aligned}
f_{0,1} &= -i\tan\Delta_{k_0}\ln\left[\frac{1+ie^{i\theta_{k_0}}\cos\Delta_{k_0}\tan\lambda}{1+ie^{-i\theta_{k_0}}\cos\Delta_{k_0}\tan\lambda}\right]\,,\\
f_{1,1} &= \tan\Delta_{k_0}\left(i\log\left[\frac{1+ie^{i\theta_{k_0}}\cos\Delta_{k_0}\tan\lambda}{1+ie^{-i\theta_{k_0}}\cos\Delta_{k_0}\tan\lambda}\right]-\frac{4\cos\Delta_{k_0}\tan\lambda\sin\theta_{k_0}}{\sin^2\theta_{k_0}+(\cos\theta_{k_0}+i\cos\Delta_{k_0}\tan\lambda)^2}\right)+\mathcal{O}\left(\sin^3(\Delta_{k_0})\right)
\end{aligned}$$

$$e^{i\theta_k}=\frac{h-e^{ik}}{\sqrt{1+h^2-2h\cos k}}\qquad \varepsilon(k)=2J\sqrt{1+h^2-2h\cos(k)}.\qquad v_k=\frac{d\epsilon(k)}{dk}$$

$$\cos\Delta_k=4\frac{hh_0-(h+h_0)\cos k+1}{\varepsilon_h(k)\varepsilon_{h_0}(k)}$$

How well does this work?





λ=0.1, ℓ=200, h_0=0, h=0.2

Summary

1. Full counting statistics for subsystems can be interesting; can be **universal** at critical points.
2. FCS is rather difficult to calculate analytically.
3. FCS in ground state of quantum critical XXZ chain
4. FCS of transverse magnetisation in TFIM in equilibrium & after quantum quenches
5. FCS after Neel quench in XXZ: interesting regime after melting of LRO
6. Hubbard model: spin-charge separation → FCS for CDW, SDW order parameters essentially Gaussian (FCS of SC order parameter in attractive case → cf. XXZ)