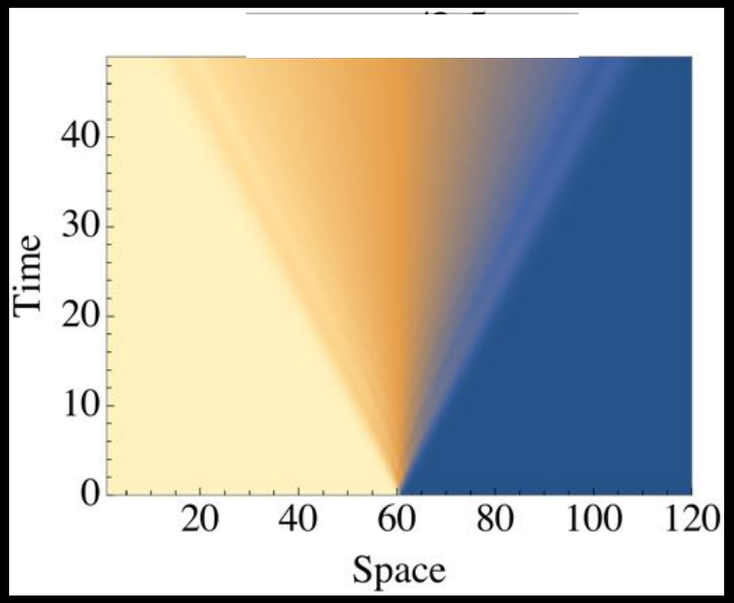
Bound states in integrable chains:
from quantum quenches to KPZ universality class



Jacopo De Nardis

In collaboration with:

Enej Ilievski	UvA
Tomaz Prosen	Ljubljana
Marko Medenjak	Ljubljana
Maurizio Fagotti	LPTMS Orsay
Bruno Bertini	Ljubljana
Mario Collura	Oxford

and more

Département de Physique
École Normale Supérieure

ENS ÉCOLE NORMALE SUPÉRIEURE
PSL RESEARCH UNIVERSITY PARIS

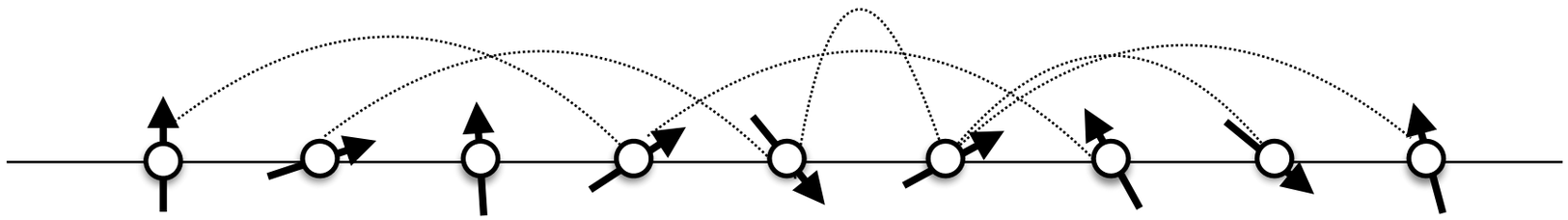
Focus on 1+1 dimension

In one dimension interactions are "strong"

strong quantum correlations

"no" mean field

collective excitations



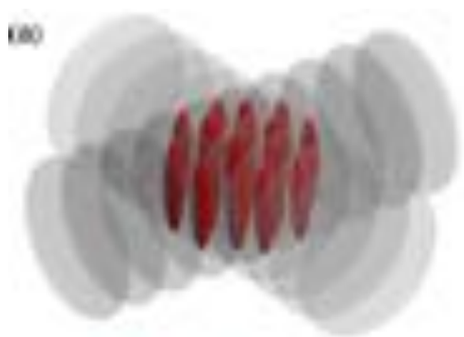
Truly Interacting many-body systems which are integrable

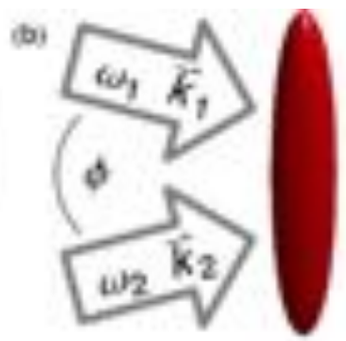
Non-perturbative collective quasi-particle excitations

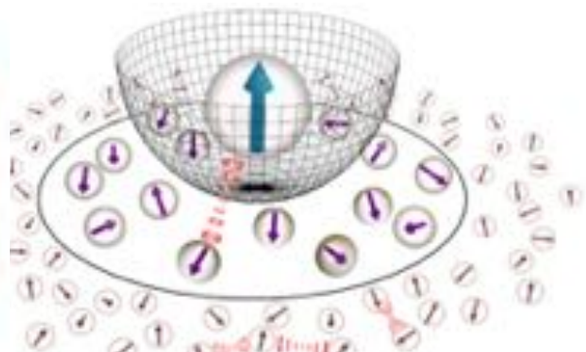
"1d" magnets

cold atoms

quantum dots



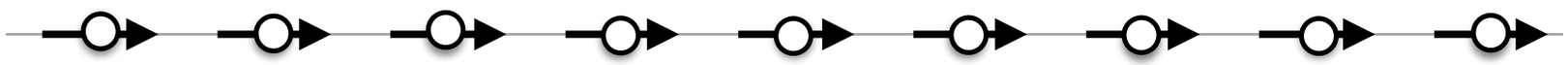




Integrable models: quantum quenches

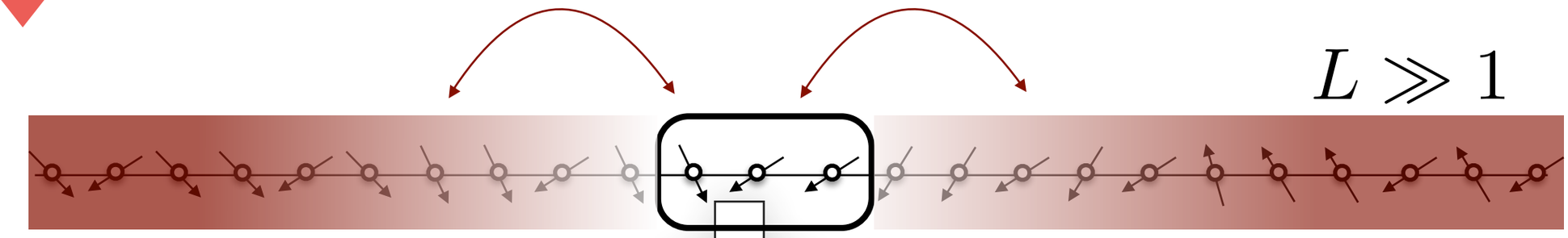
Quantum Quench

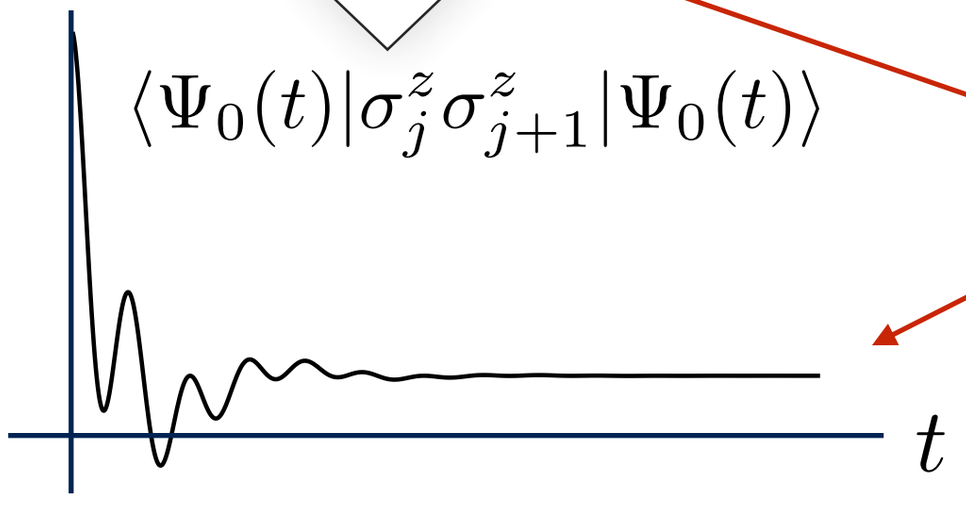
$|\Psi_0\rangle$



e^{-itH}

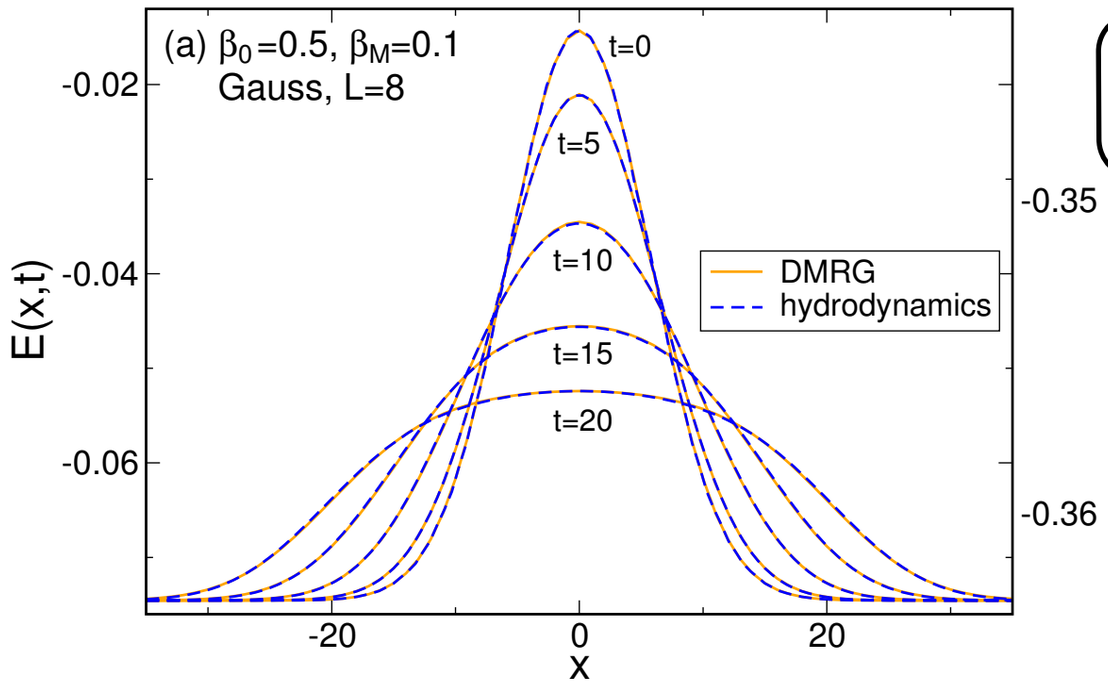
How do steady states emerge locally in closed systems?





$\hat{\rho}_{\text{steady}}$

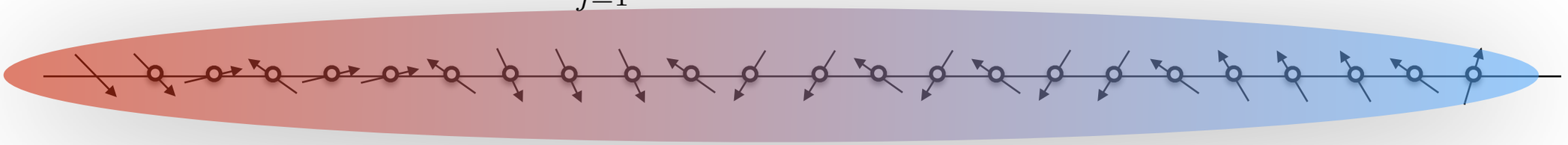
Integrable models: emergent hydrodynamics



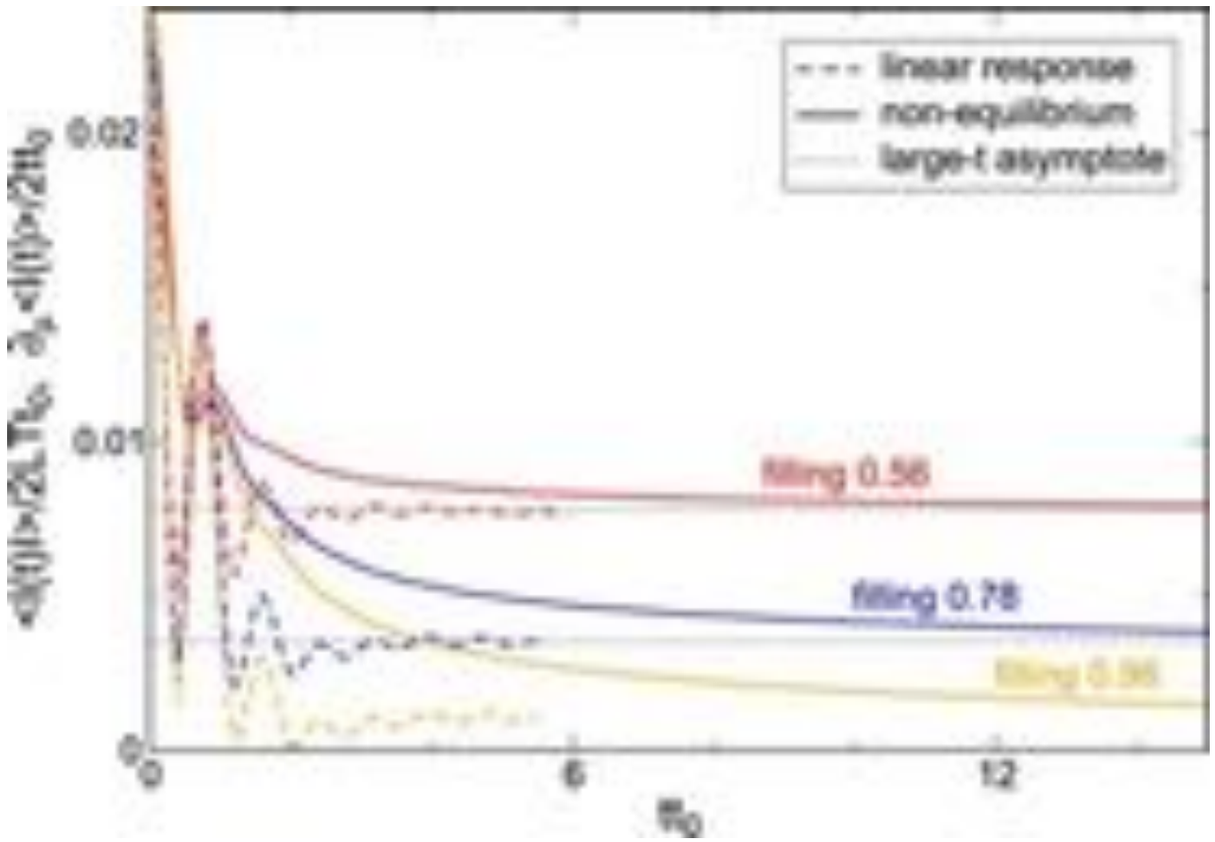
Bulchandani et al,
Phys. Rev. Lett. (2017)

$$\partial_t n_{x,t} = \partial_x \mathcal{J}(n_{x,t})$$

$$\hat{H}_{\mathrm{XXZ}} = \sum_{j=1}^{L} \left(\hat{S}_j^x \hat{S}_{j+1}^x + \hat{S}_j^y \hat{S}_{j+1}^y + \Delta \hat{S}_j^z \hat{S}_{j+1}^z \right)$$



Integrable models: transport (and diffusion)



C Karrasch, New. J. Physics (2017)

$$D(\beta) = \lim_{T\to\infty} \frac{\beta}{\chi(\beta)} \int dx \int_0^T dt \left\langle \hat{j}(x,t)\hat{j}(0,0) \right\rangle$$

$$\langle \hat{j}(x,t)\hat{j}(0,0)\rangle \to \langle \hat{j}(x,t)\hat{j}(0,0)\rangle - 2\mathcal{D}$$

Integrable systems out of equilibrium

Intro on integrable models and Bethe states

1. Quantum quenches and GGE

The role of the bound states to determine the steady state

2. Hydrodynamics of integrable models and transport theory

The bi-partite quantum quench

Drude weights

Divergence of diffusion constant in symmetric models

3. KPZ fluctuations and memory

The role of bound states in the Lieb-Liniger model

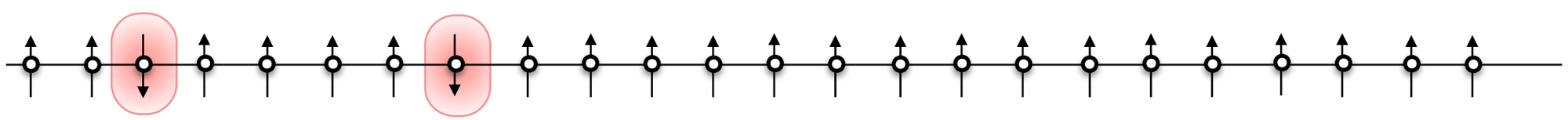
The Bethe wave function

$$\Psi(\{x_j\}) \sim \sum_P \left(\prod_{j=1}^{N} e^{i x_j k_{P_j}} \right) \left(\prod_{i<j} e^{-iS(k_{P_i}, k_{P_j})} \right)$$

$$\hat{H}_{\mathrm{XXZ}} = \sum_{j=1}^{L} \left(\hat{S}^x_j \hat{S}^x_{j+1} + \hat{S}^y_j \hat{S}^y_{j+1} + \Delta \hat{S}^z_j \hat{S}^z_{j+1} \right)$$

XXZ spin chain

$$|E\rangle = \sum_{x_1,\ldots,x_N} \Psi(\{x_j\}) \sigma^-_{x_1} \ldots \sigma^-_{x_N} |\uparrow, \ldots, \uparrow\rangle$$



Bound states

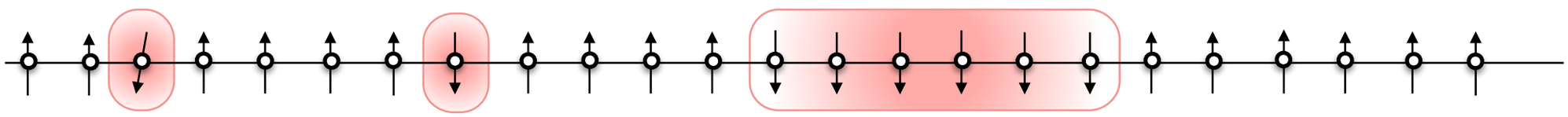
$$\Psi(\{x_j\}) \sim \sum_P \left(\prod_{j=1}^{N} e^{i x_j k_{P_j}} \right) \left(\prod_{i<j} e^{-iS(k_{P_i}, k_{P_j})} \right)$$

$$e^{ik_i L} = \prod_{i \neq j} e^{-iS(k_i, k_j)}$$

Bethe Equations

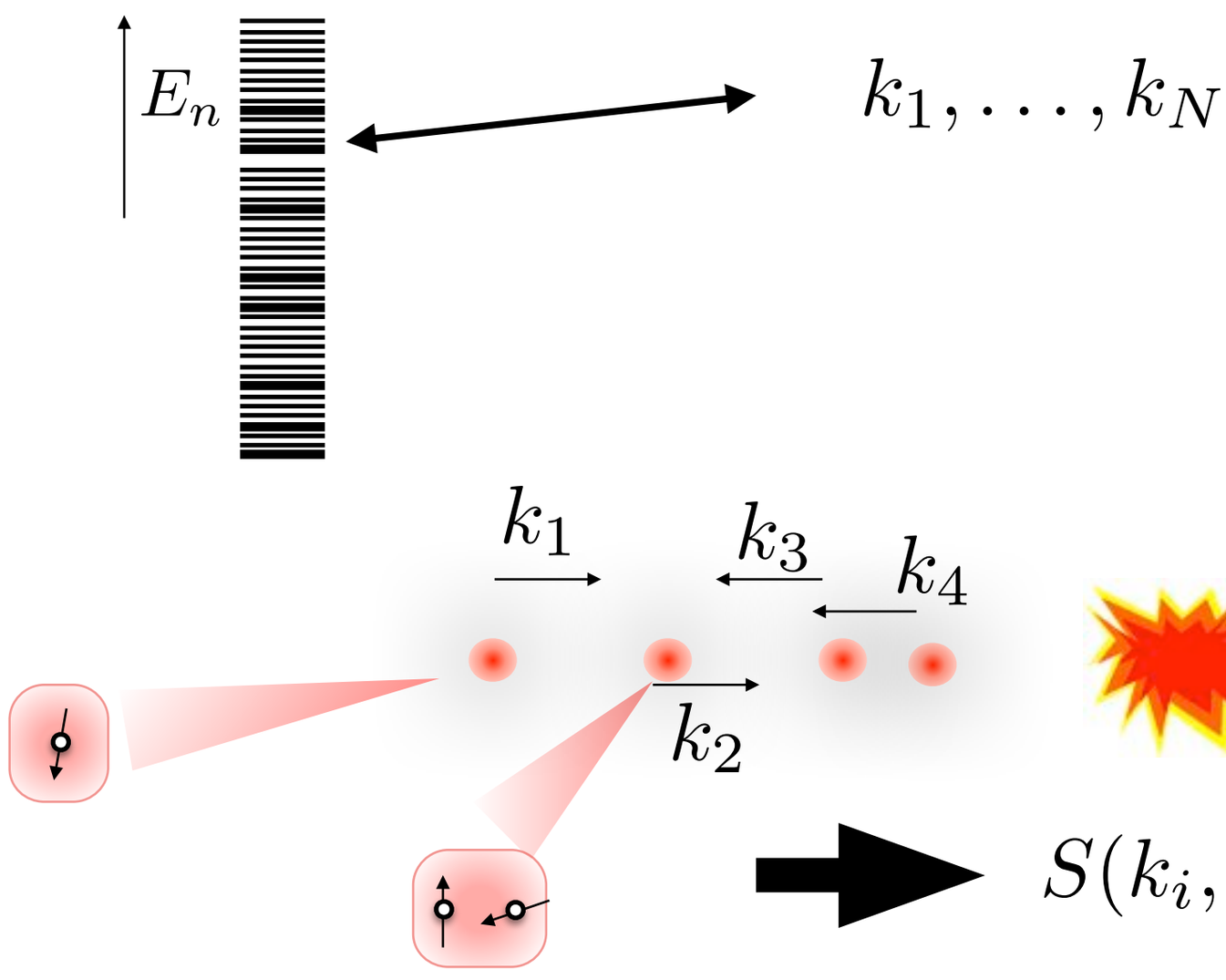
Complex solutions: bound states

$$\Psi \sim e^{isk/2 \sum_{j=1}^{s} x_j - \kappa \sum_{i<j=1}^{s} |x_i - x_j|}$$

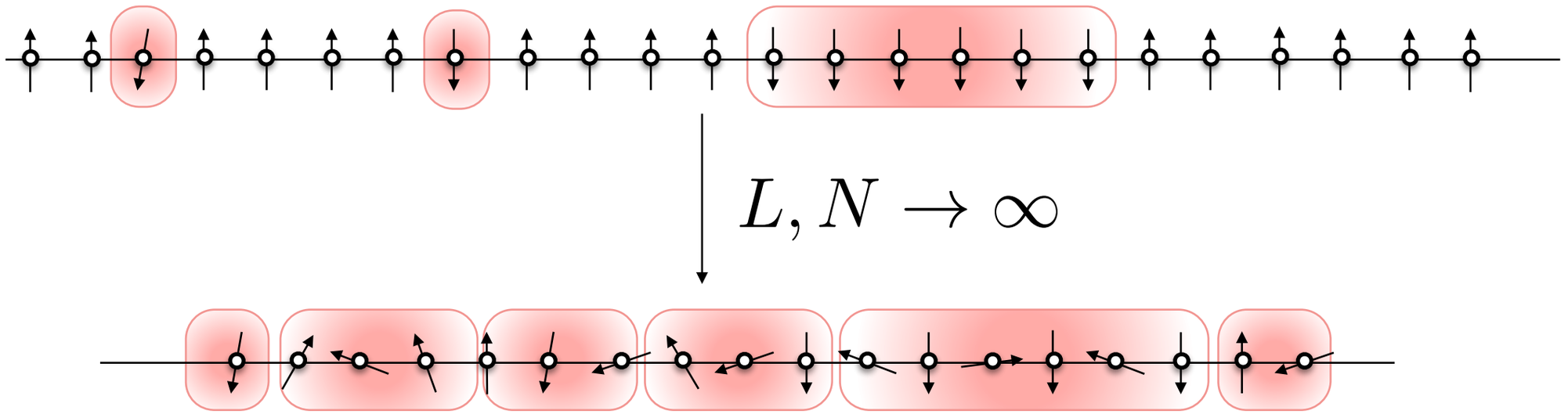


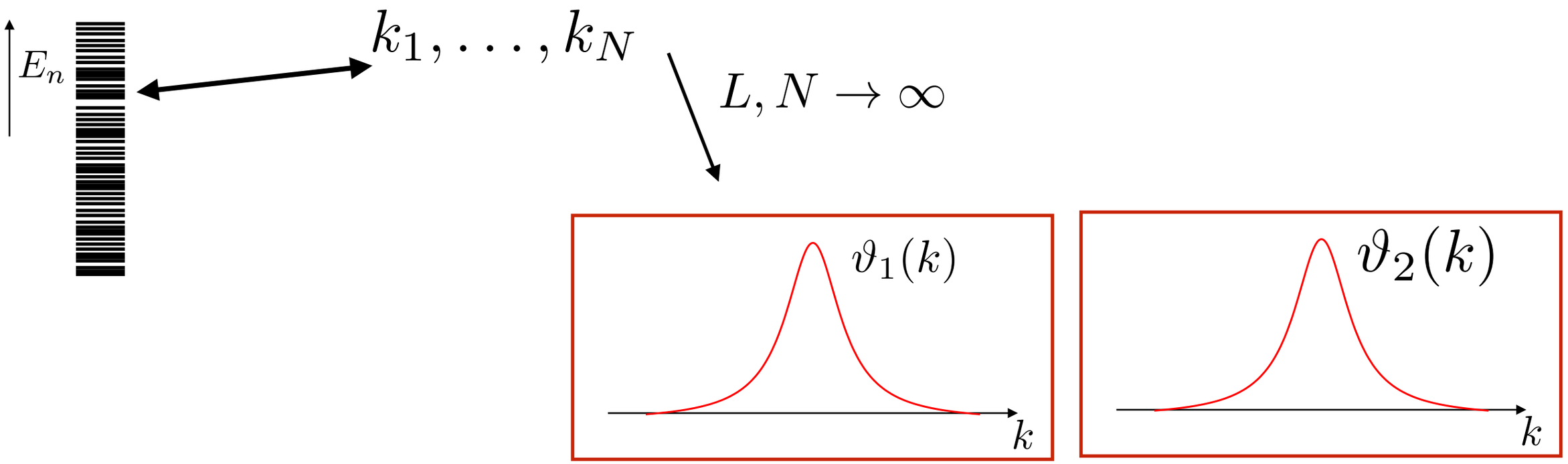
Eigenstates and particles

Integrable models:
quasi-particle description of the eigenstates



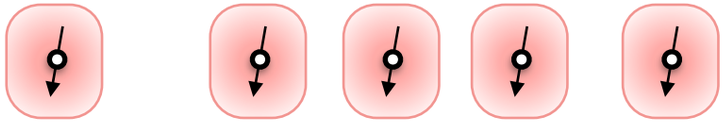
The thermodynamic limit

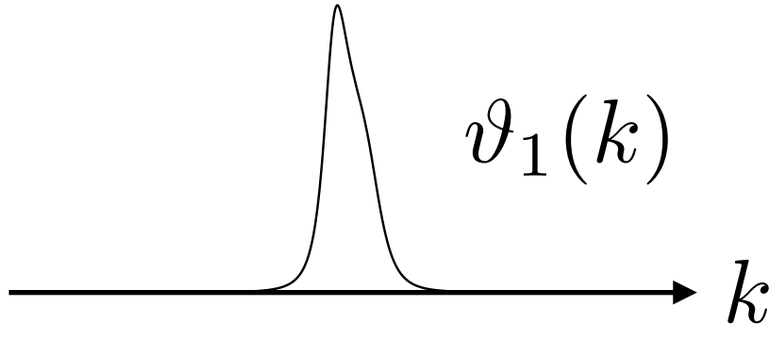




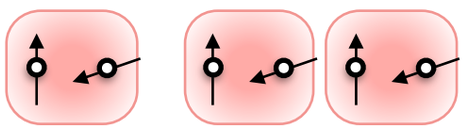
Particles types and symmetry

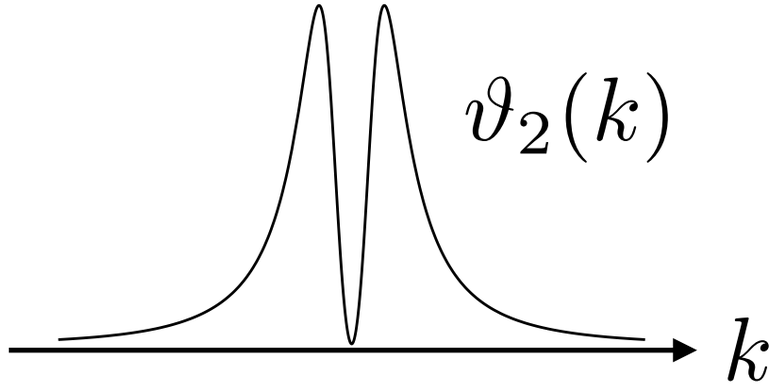
$$H_{XXX} = \sum_{j=1}^{L} \hat{S}_j \cdot \hat{S}_{j+1}$$ XXX spin chain $SU(2)$



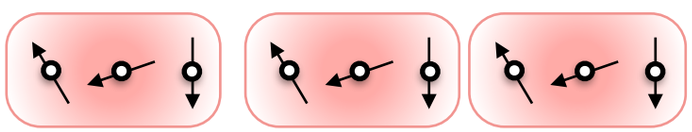


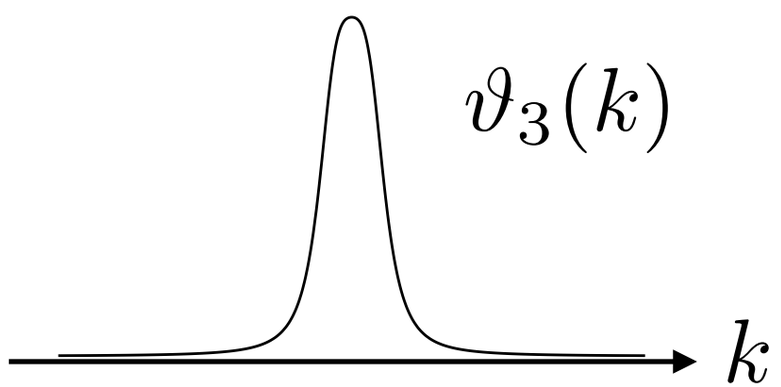
$m_1^{\text{bare}} = 1$





$m_2^{\text{bare}} = 2$

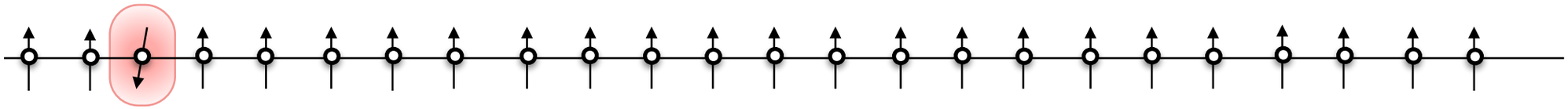




$m_3^{\text{bare}} = 3$

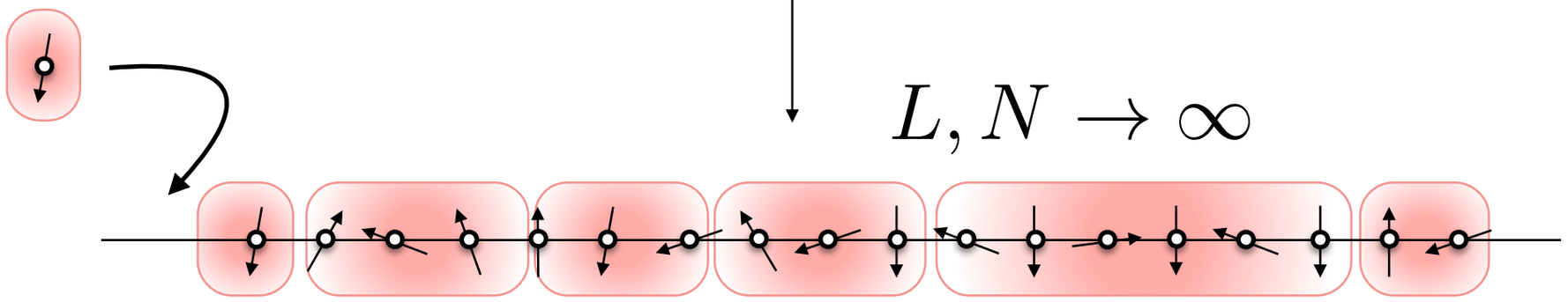
.....

Excitations and dressing



$\varepsilon(k) \qquad m_1^{\mathrm{bare}}$

$L, N \to \infty$



$\varepsilon^{\mathrm{dr}}(k^{\mathrm{dr}}) \;\; m_1^{\mathrm{dr}}$

$$q^{\mathrm{dr}} = (1 - T\vartheta)^{-1} q$$

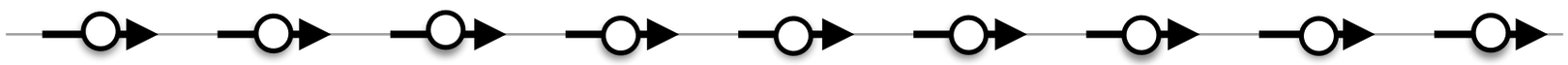
$$T(k, k') = \frac{1}{2\pi} \partial_{k'} S(k, k')$$

Quantum Quenches and GGE

Out-of-equilibrium isolated dynamics

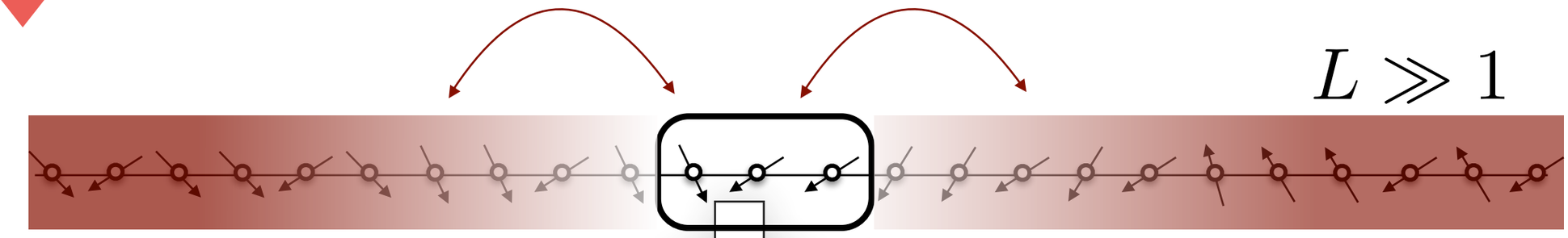
Quantum Quench

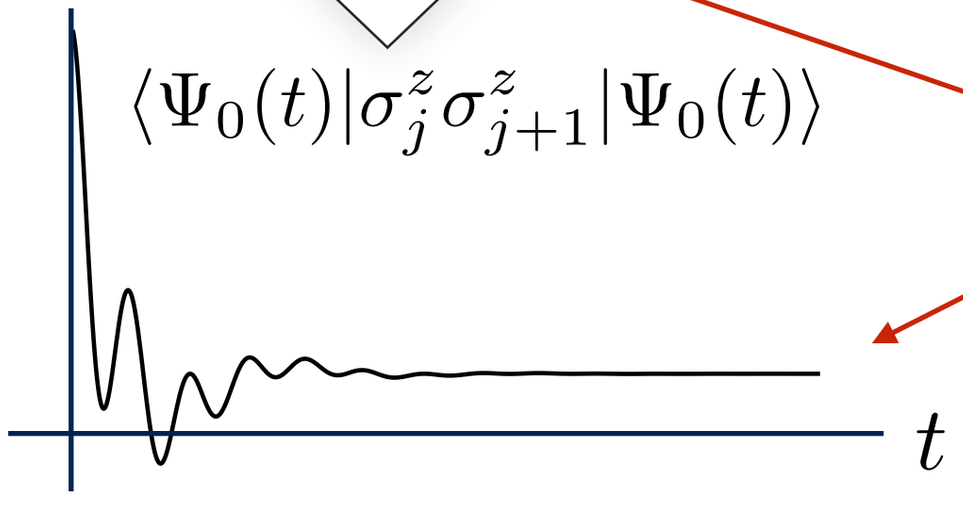
$|\Psi_0\rangle$



e^{-itH}

How do steady states emerge locally in closed systems?





$\hat{\rho}_{\text{steady}}$

J. Stat. Mech, Special Edition, Non-equilibrium (2016)

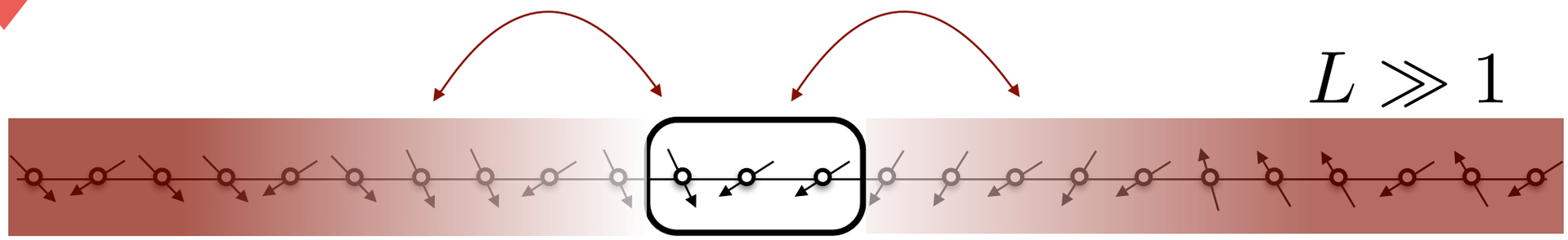
The Generalized Gibbs ensemble

Rigol, Dunjko, Olshanii, Nature (2008)
Rigol et al, Phys. Rev. Lett (2007)

$|\Psi_0\rangle$

e^{-itH}

How to characterize the steady state given the initial one?



$L \gg 1$

GGE

$$\hat{\rho}_{\text{steady}} = Z^{-1} e^{-\sum_n \beta_n Q_n}$$

$$[H, Q_n] = 0 \qquad \langle \Psi_0 | \hat{Q}_m | \Psi_0 \rangle = \text{Tr}[\hat{\rho}_{\text{steady}} Q_m]$$

The GGE in interacting systems

$$|\Psi_0\rangle = \prod_j |\uparrow\downarrow\rangle_{j,j+1} - |\downarrow\uparrow\rangle_{j,j+1}$$

$$\hat{H}_{\mathrm{XXZ}} = \sum_{j=1}^{L} \left(\hat{S}^x_j \hat{S}^x_{j+1} + \hat{S}^y_j \hat{S}^y_{j+1} + \Delta \hat{S}^z_j \hat{S}^z_{j+1} \right)$$

PRL 113, 117203 (2014) PHYSICAL REVIEW LETTERS week ending 12 SEPTEMBER 2014

Correlations after Quantum Quenches in the *XXZ* Spin Chain: Failure of the Generalized Gibbs Ensemble

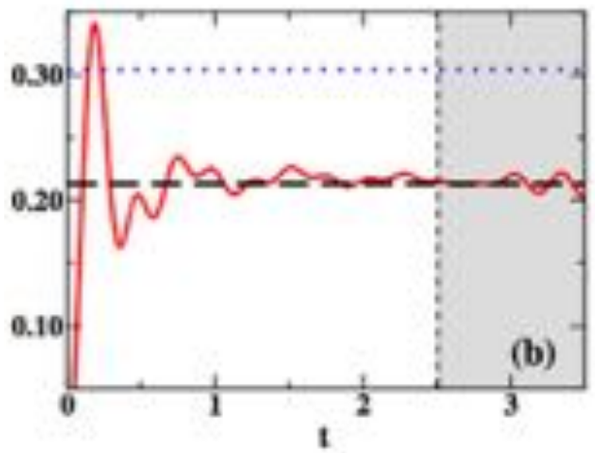
B. Pozsgay,[1] M. Mestyán,[1] M. A. Werner,[2] M. Kormos,[1] G. Zaránd,[2] and G. Takács[1,*]
[1]*MTA-BME "Momentum" Statistical Field Theory Research Group, Department of Theoretical Physics, Budapest University of Technology and Economics, 1111 Budapest, Budafoki út 8, Hungary*
[2]*MTA-BME "Momentum" Exotic Quantum Phases Research Group, Department of Theoretical Physics, Budapest University of Technology and Economics, 1111 Budapest, Budafoki út 8, Hungary*
(Received 15 May 2014; published 9 September 2014)

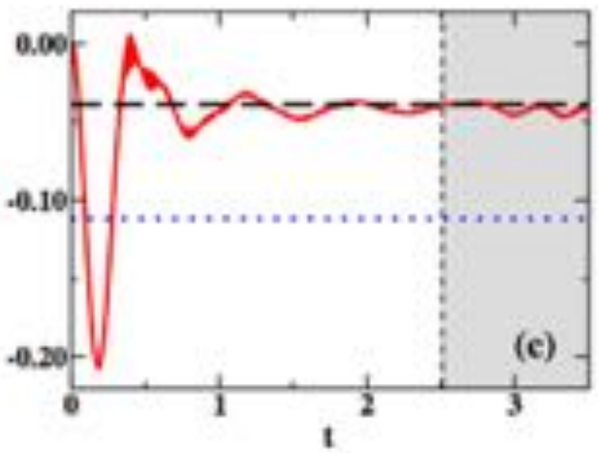
PRL 113, 117202 (2014) PHYSICAL REVIEW LETTERS week ending 12 SEPTEMBER 2014

Quenching the Anisotropic Heisenberg Chain: Exact Solution and Generalized Gibbs Ensemble Predictions

B. Wouters,[1] J. De Nardis,[1] M. Brockmann,[1] D. Fioretto,[1] M. Rigol,[2] and J.-S. Caux[1]
[1]*Institute for Theoretical Physics, University of Amsterdam, Science Park 904, Postbus 94485, 1090 GL Amsterdam, Netherlands*
[2]*Department of Physics, The Pennsylvania State University, University Park, Pennsylvania 16802, USA*
(Received 15 May 2014; revised manuscript received 22 July 2014; published 9 September 2014)

$\langle(\sigma^z_1\sigma^z_2)(t)\rangle$

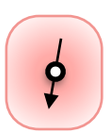




$\langle(\sigma^z_1\sigma^z_3)(t)\rangle$

$$\hat{\rho}_{\mathrm{steady}} = Z^{-1} e^{-\sum_n \beta_n Q_n}$$

$[H, Q_n] = 0$ $\quad \langle\Psi_0|\hat{Q}_m|\Psi_0\rangle = \mathrm{Tr}[\hat{\rho}_{\mathrm{steady}} Q_m]$ ⟷



The GGE in interacting systems

$$|\Psi_0\rangle = \prod_j |\uparrow\downarrow\rangle_{j,j+1} - |\downarrow\uparrow\rangle_{j,j+1}$$

$$\hat{H}_{\mathrm{XXZ}} = \sum_{j=1}^{L} \left(\hat{S}^x_j \hat{S}^x_{j+1} + \hat{S}^y_j \hat{S}^y_{j+1} + \Delta \hat{S}^z_j \hat{S}^z_{j+1} \right)$$

PRL 115, 157201 (2015) PHYSICAL REVIEW LETTERS week ending 9 OCTOBER 2015

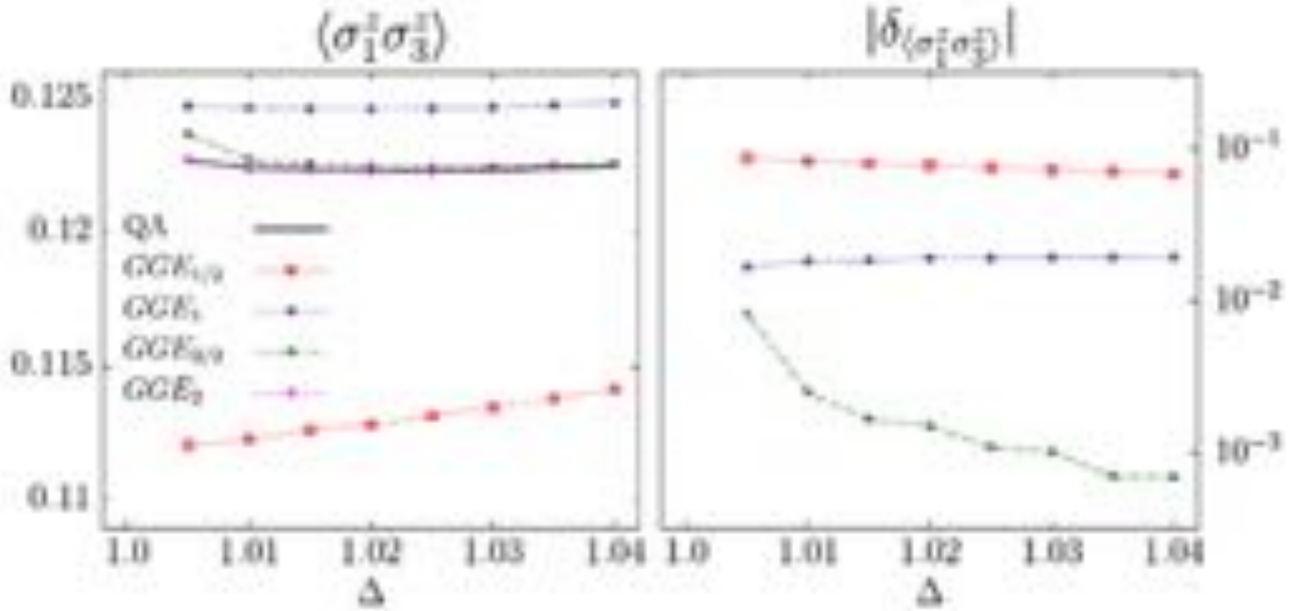
Complete Generalized Gibbs Ensembles in an Interacting Theory

E. Ilievski,[1] J. De Nardis,[1] B. Wouters,[1] J.-S. Caux,[1] F. H. L. Essler,[2] and T. Prosen[3]
[1]*Institute for Theoretical Physics, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, The Netherlands*
[2]*The Rudolf Peierls Centre for Theoretical Physics, University of Oxford, Oxford OX1 3NP, United Kingdom*
[3]*Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia*

(Received 21 July 2015; published 7 October 2015)

Ilievski, Medenjak, Prosen, Phys. Rev. Lett. (2015)

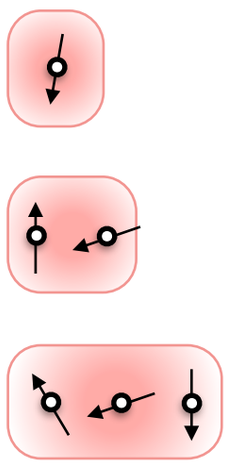
Quasi-local charges



Ilievski, Quinn, Caux, Phys. Rev. B. (2017)

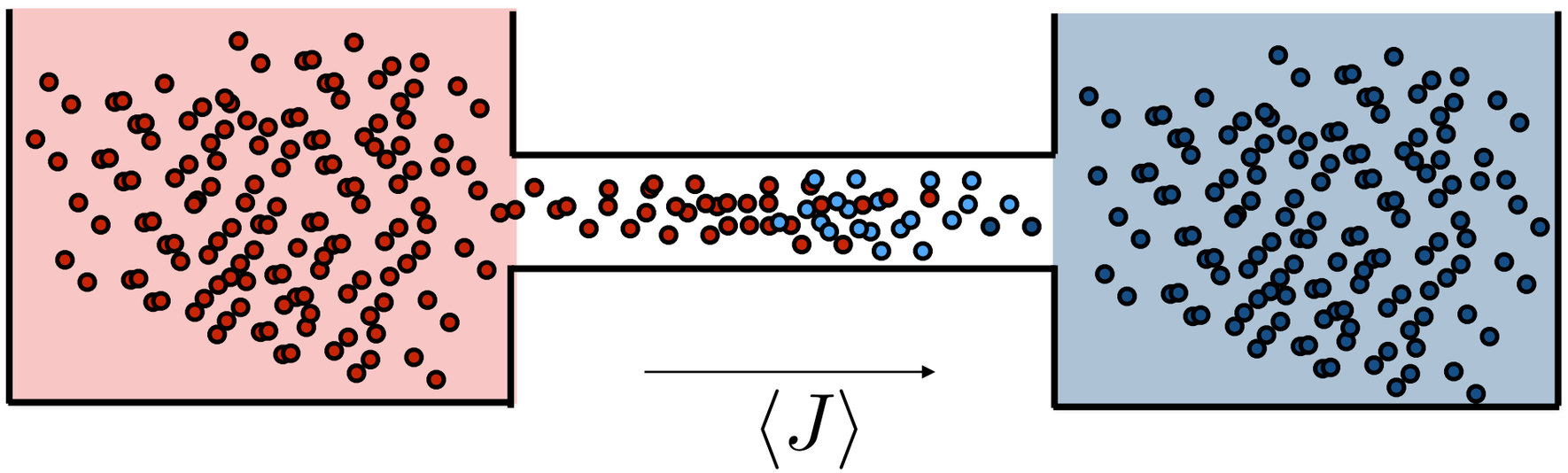
$$\hat{\rho}_{\mathrm{steady}} = Z^{-1} e^{-\sum_n \beta_n Q_n}$$

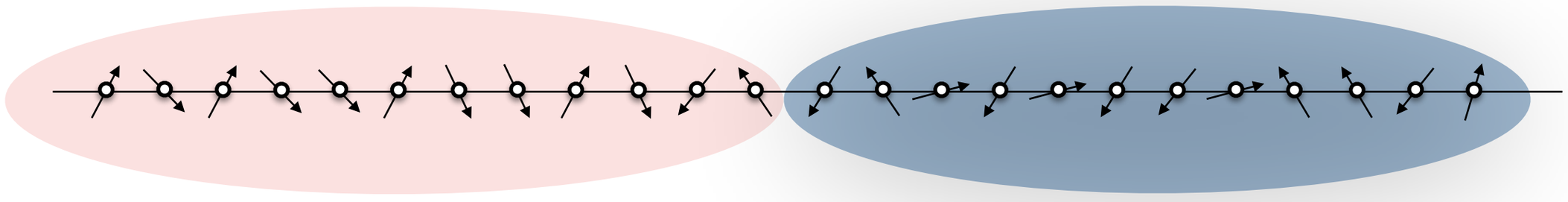
$$[H, Q_n] = 0 \qquad \langle\Psi_0|\hat{Q}_m|\Psi_0\rangle = \mathrm{Tr}[\hat{\rho}_{\mathrm{steady}} Q_m]$$



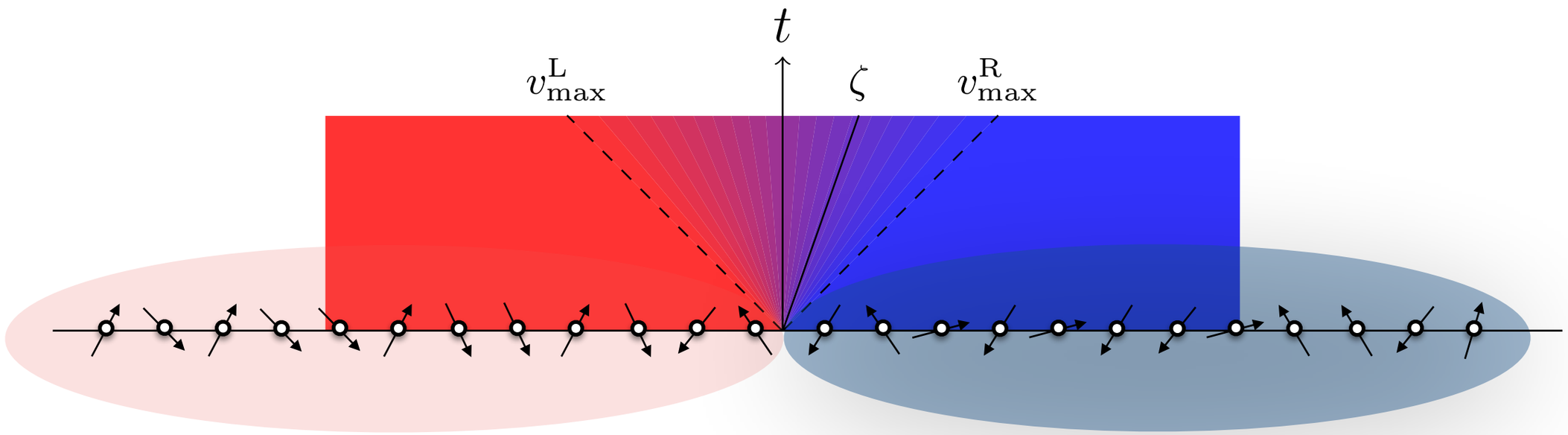
Hydrodynamics and transport

Bi-partite (inhomogenous) quenches

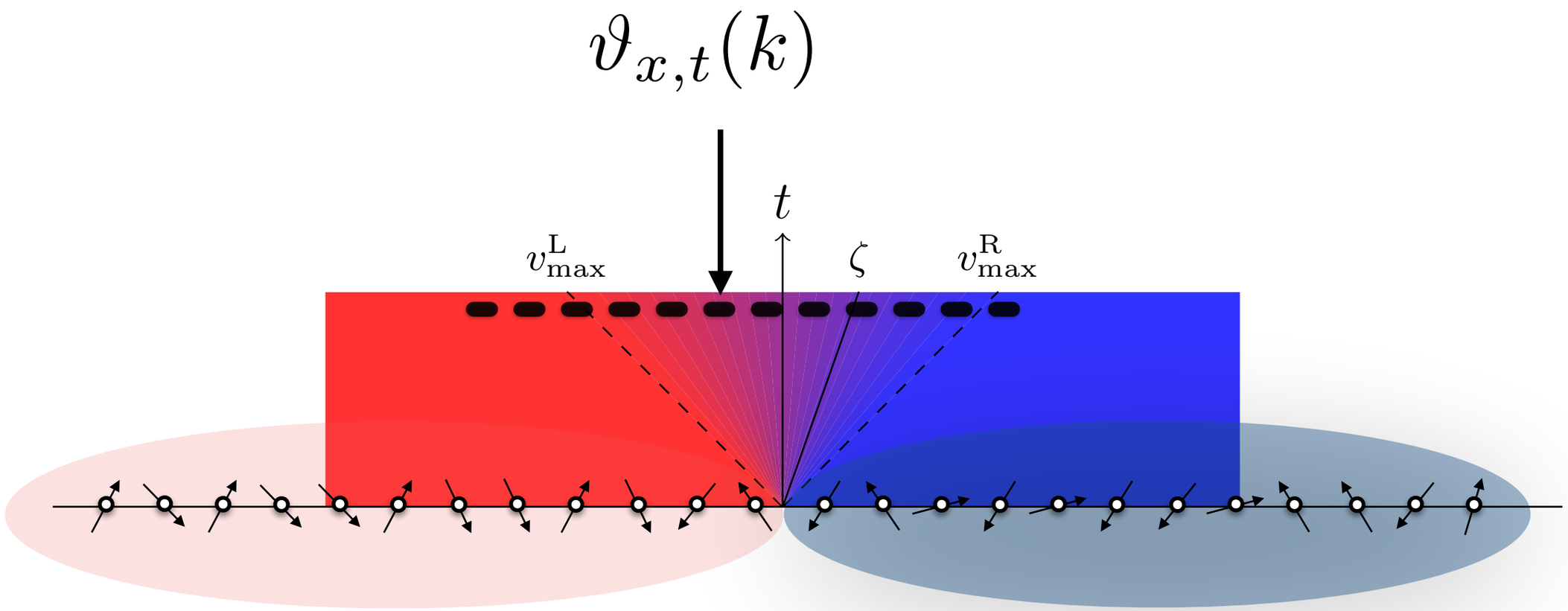




Bi-partite (inhomogenous) quenches

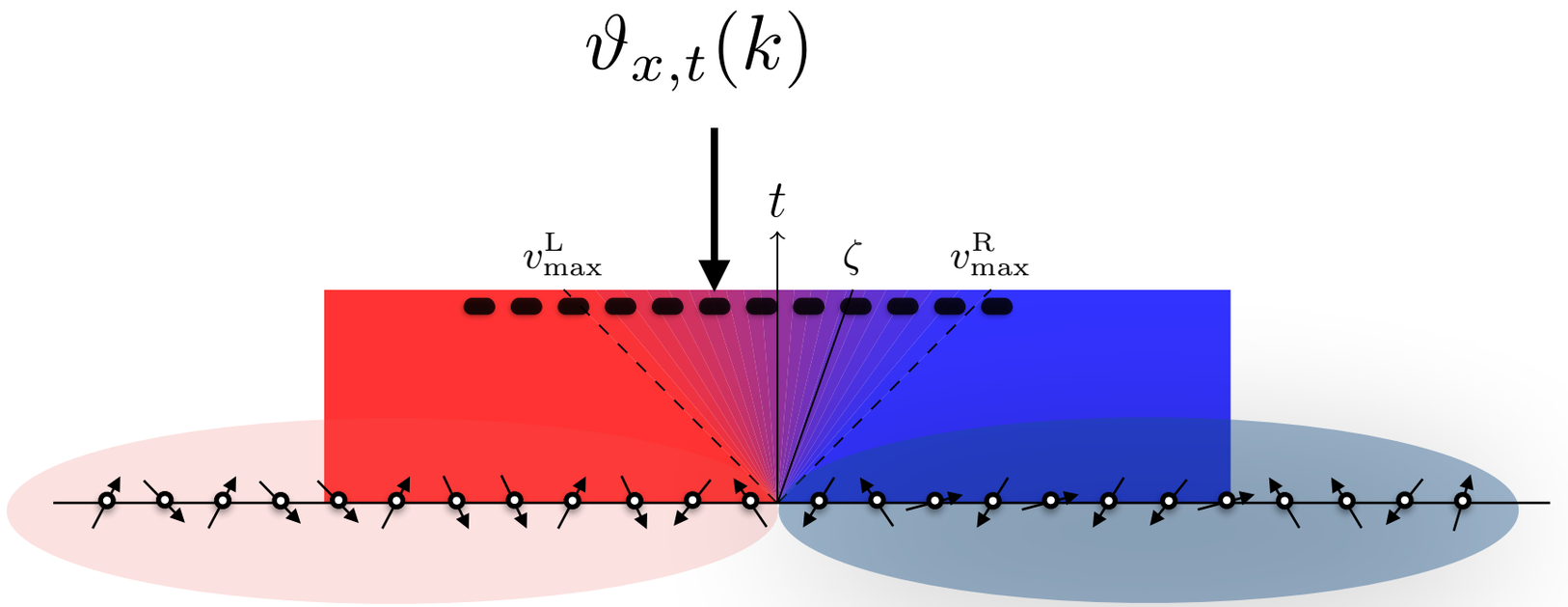


Large-scale hydrodynamic description



$$\partial_t \vartheta_{x,t}(k) = J(\vartheta_{x,t}, \partial_x \vartheta_{x,t}(k))$$

Large-scale hydrodynamic description



$$\partial_t \vartheta_{x,t}(k) = J(\vartheta_{x,t}, \partial_x \vartheta_{x,t}(k))$$

Selected for a Viewpoint in *Physics*
PHYSICAL REVIEW X **6,** 041065 (2016)

Emergent Hydrodynamics in Integrable Quantum Systems Out of Equilibrium

Olalla A. Castro-Alvaredo,[1] Benjamin Doyon,[2] and Takato Yoshimura[2]
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Transport in Out-of-Equilibrium *XXZ* Chains: Exact Profiles of Charges and Currents

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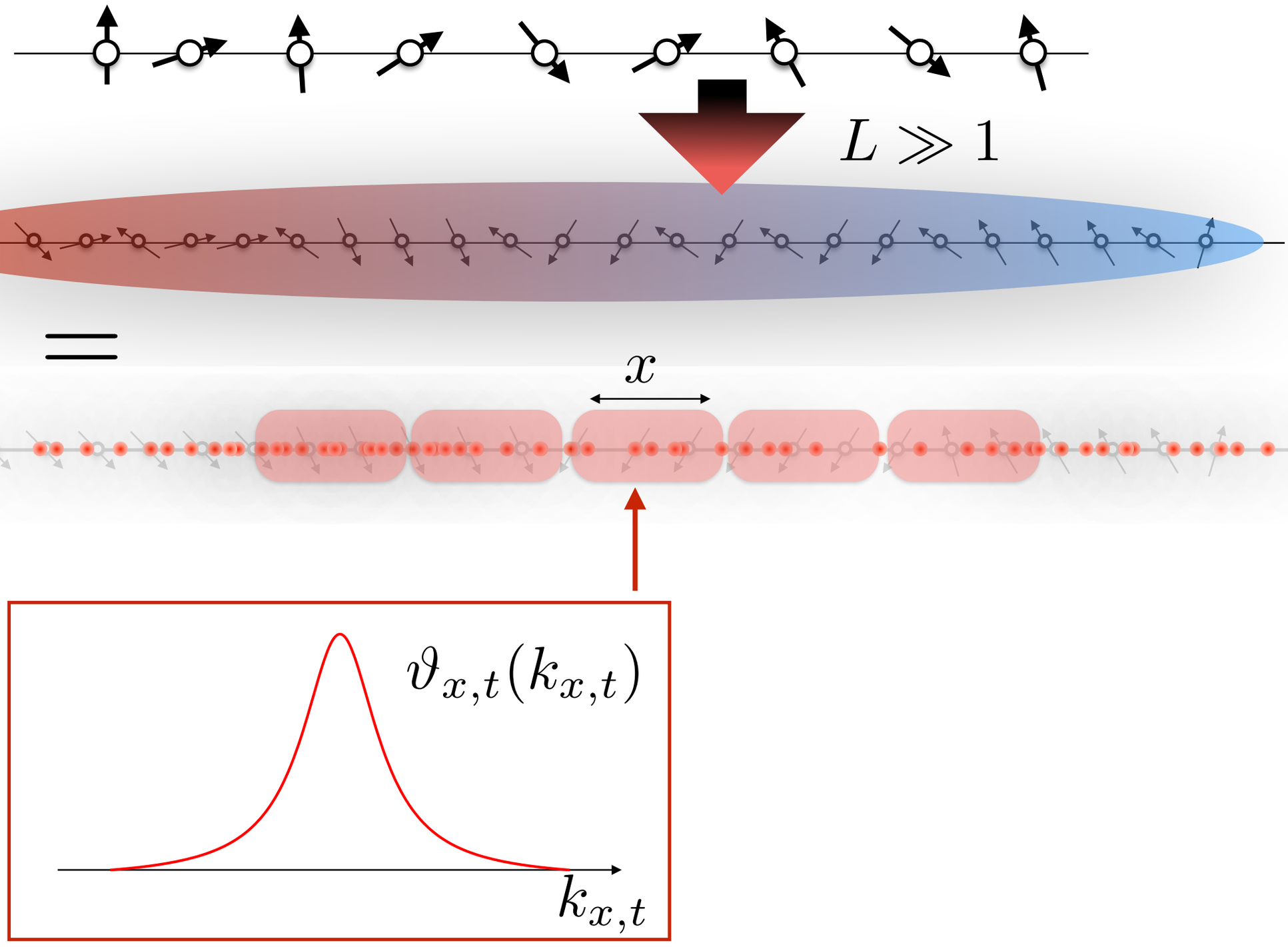
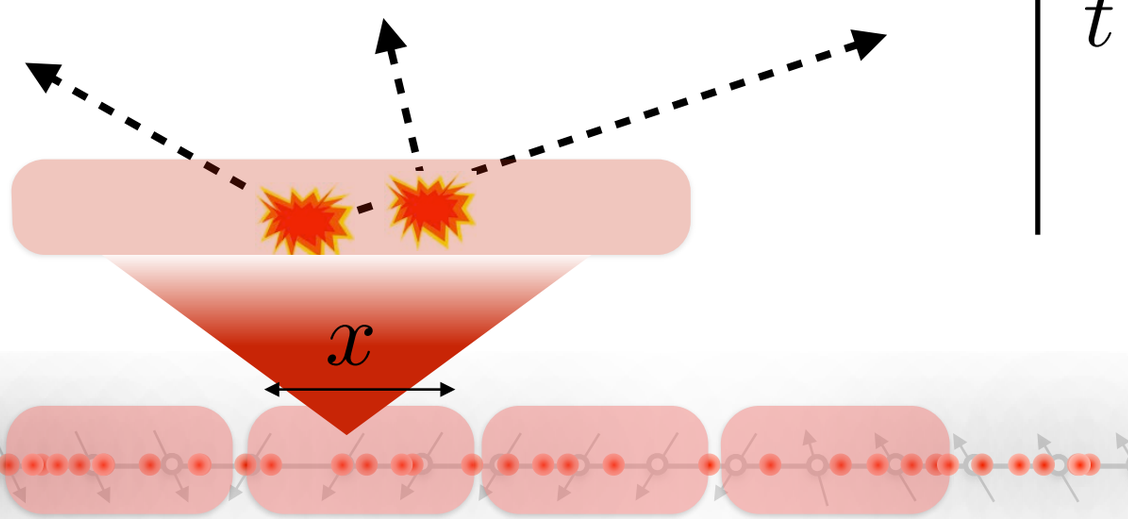
Large-scale hydrodynamic description



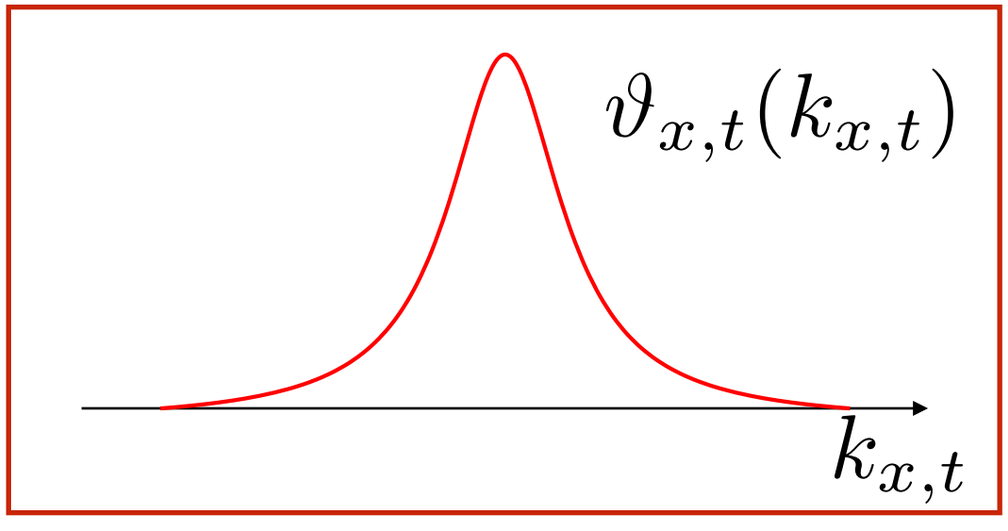
Large-scale hydrodynamic description

$$v^{\mathrm{eff}}_{x,t}(k) = v^{\mathrm{bare}}(k) + \int dk' \vartheta_{x,t}(k') S(k,k') (v^{\mathrm{eff}}(k) - v^{\mathrm{eff}}(k'))$$



$$v^{\mathrm{eff}}_{x,t}(k^{\mathrm{dr}}) = \frac{\partial \varepsilon^{\mathrm{dr}}(k^{\mathrm{dr}})}{\partial k^{\mathrm{dr}}}$$

Bonnes, Essler, Läuchli, Phys. Rev. Lett. (2014)



$$\Big(\partial_t + v^{\mathrm{eff}}[\vartheta_{x,t}]\partial_x\Big)\vartheta_{x,t}(k_{x,t}) = 0$$

Exact semi-classical description in the limit

$t \to \infty$

with x/t finite

Emergent hydrodynamic description

A classical hydrodynamics description of a quantum systems at large times

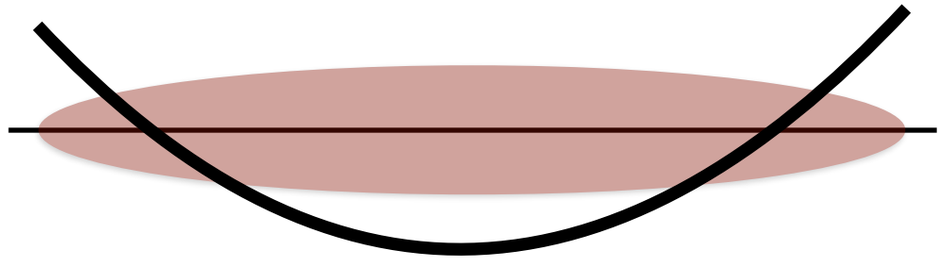
Doyon, Yoshimura, SciPost Phys. (2017)

Cao, Bulchandani, Moore Phys. Rev. Lett. (2018)

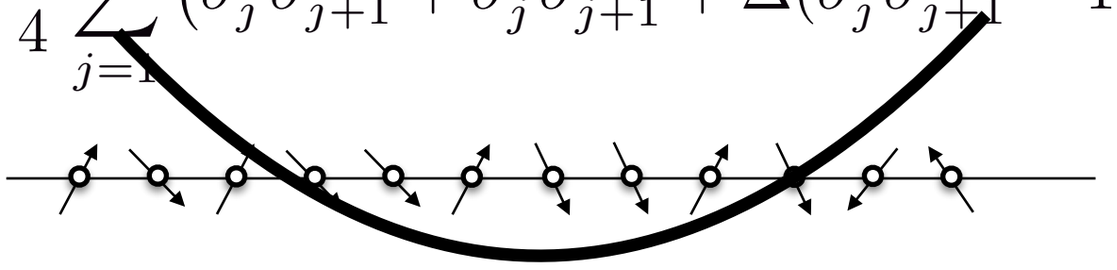
$$\Big(\partial_t + v^{\rm eff}[\vartheta_{x,t}]\partial_x\Big)\vartheta_{x,t}(k_{x,t}) = 0 + f[\vartheta_{x,t}]\partial_x V(x)$$

$x \gg 1$ $\qquad$ $t \gg 1$

$$H = -\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + 2c\sum_{j>k}\delta(x_j - x_k)$$



$$H = \frac{J}{4}\sum_{j=1}^{N}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta(\sigma_j^z\sigma_{j+1}^z - 1)\right)$$

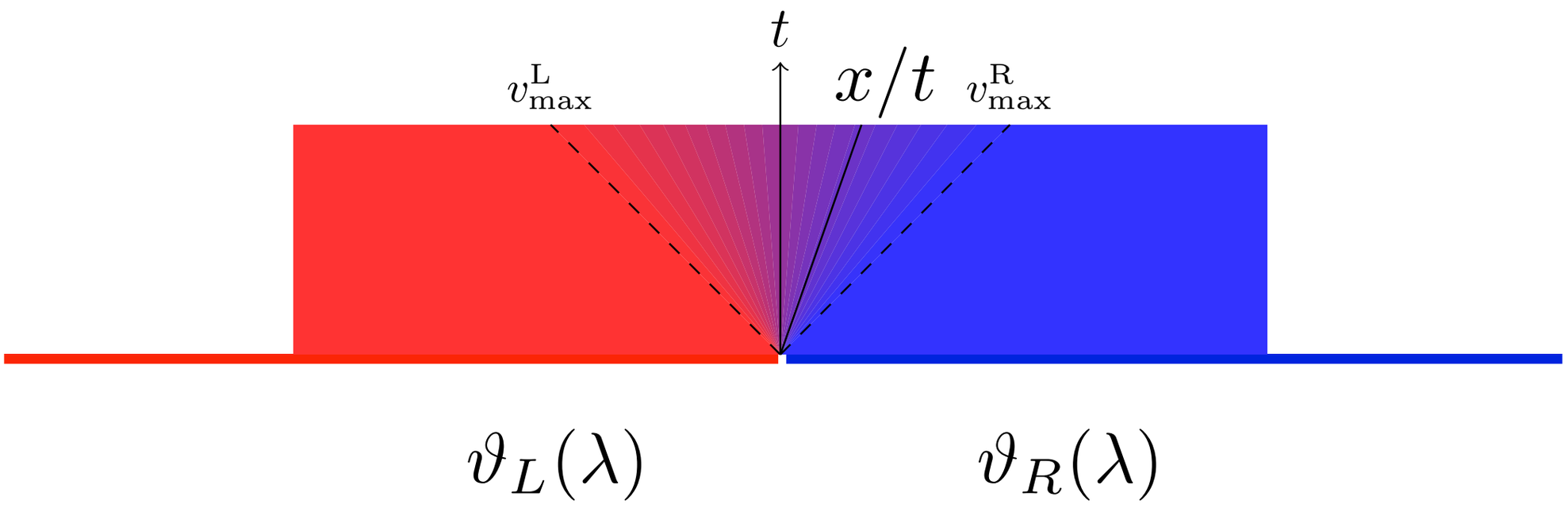


Emergent hydrodynamic description

$$(\partial_t + v_{x,t}\partial_x)\vartheta_{x,t} = 0$$

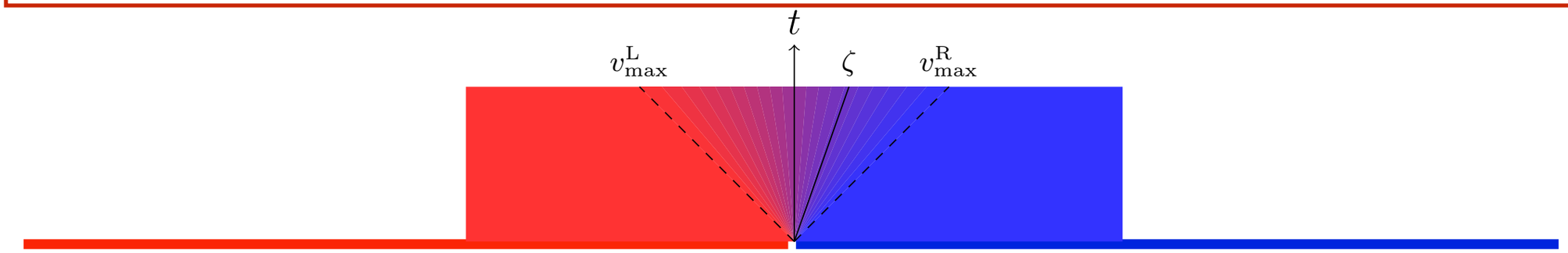
$$t \to \infty \qquad \vartheta_{x,t} \to \vartheta_{x/t}$$

$$\vartheta_{x/t}(k) = \begin{cases} \vartheta_L(k) \text{ if } v_{x/t}(k) \geq x/t \\ \vartheta_R(k) \text{ if } v_{x/t}(k) < x/t \end{cases}$$

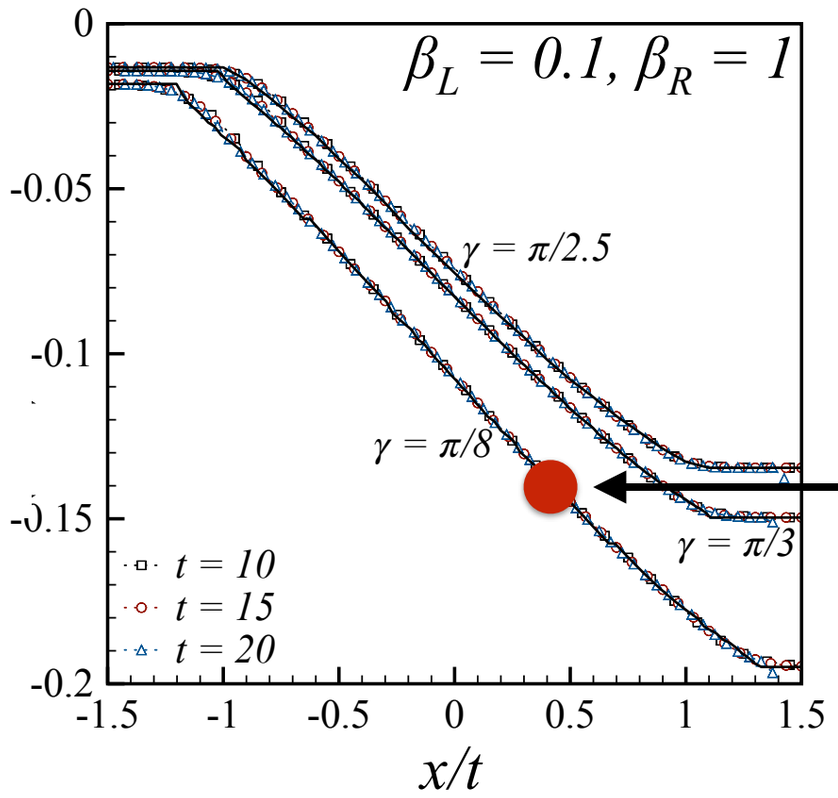


Emergent hydrodynamic description

$$H = \frac{J}{4}\sum_{j=1}^{N}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta(\sigma_j^z\sigma_{j+1}^z - 1)\right) \qquad \Delta = \cos\gamma$$



$e^{-\beta_L(H-\mu_L)}/Z$ $\qquad$ $e^{-\beta_R(H-\mu_R)}/Z$

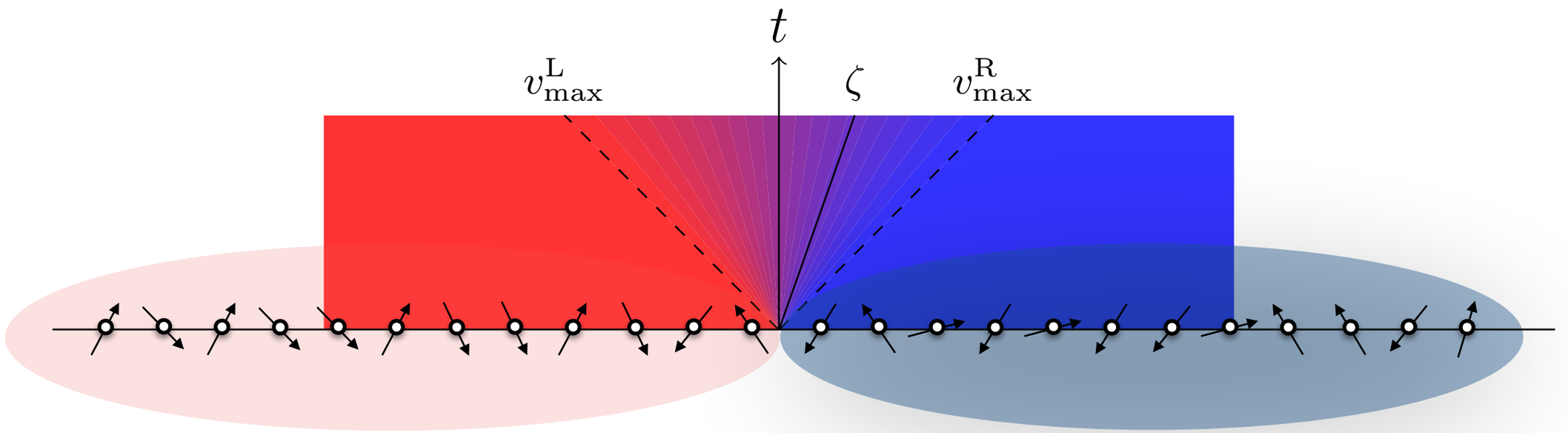


$$\vartheta_{x,t}(k) \to \vartheta_{x/t}(k)$$

$\vartheta_{x/t}(k)$

$$\langle e \rangle_{x/t} = \int dk\ \vartheta_{x/t}(k) e_0(k)$$

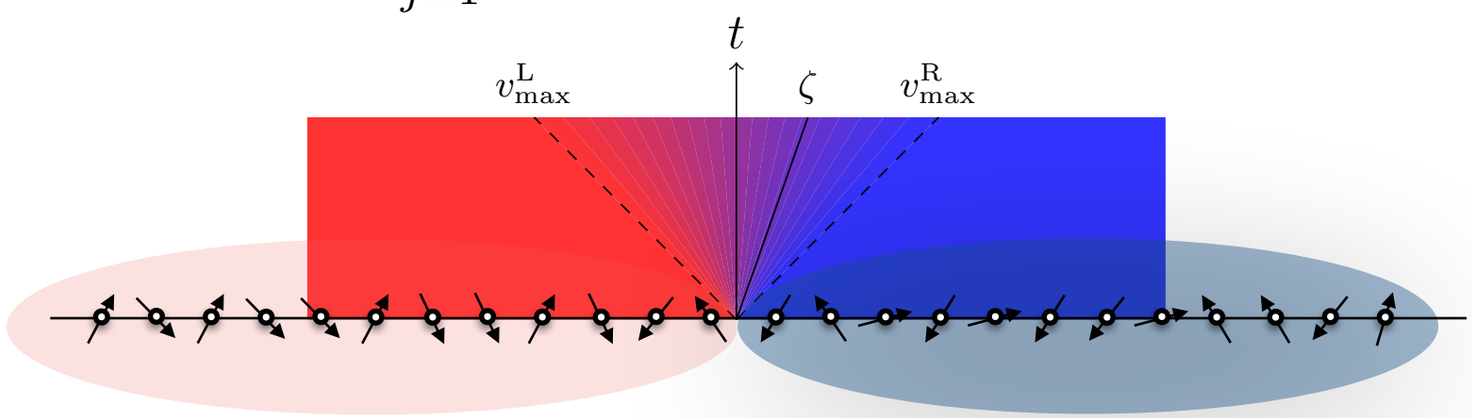
Drude weights: DC conductivity



$$\sigma = \lim_{\delta h \to 0} \lim_{t \to \infty} \frac{1}{\delta h} \int dx \langle j(x,t) \rangle \quad \sim 2t\mathcal{D}$$

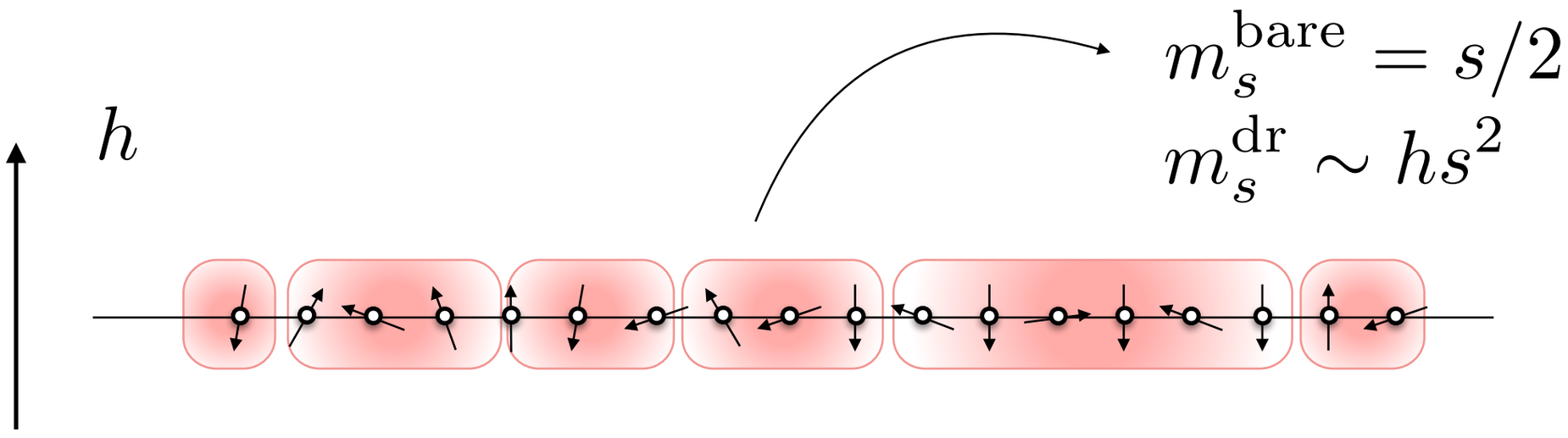
Drude weights: DC conductivity

$$\hat{H}_{\mathrm{XXZ}} = \sum_{j=1}^{L} \left(\hat{S}_j^x \hat{S}_{j+1}^x + \hat{S}_j^y \hat{S}_{j+1}^y + \Delta \hat{S}_j^z \hat{S}_{j+1}^z \right)$$



$$\mathcal{D}^{(q)} = \frac{\beta}{2} \sum_A \int dk \mathcal{D}_A(k) [q_A^{\mathrm{dr}}(k)]^2$$

$$\mathcal{D}_A(k) = \vartheta_A(k)(1 - \vartheta_A(k))(v_A^{\mathrm{eff}}(k))^2$$

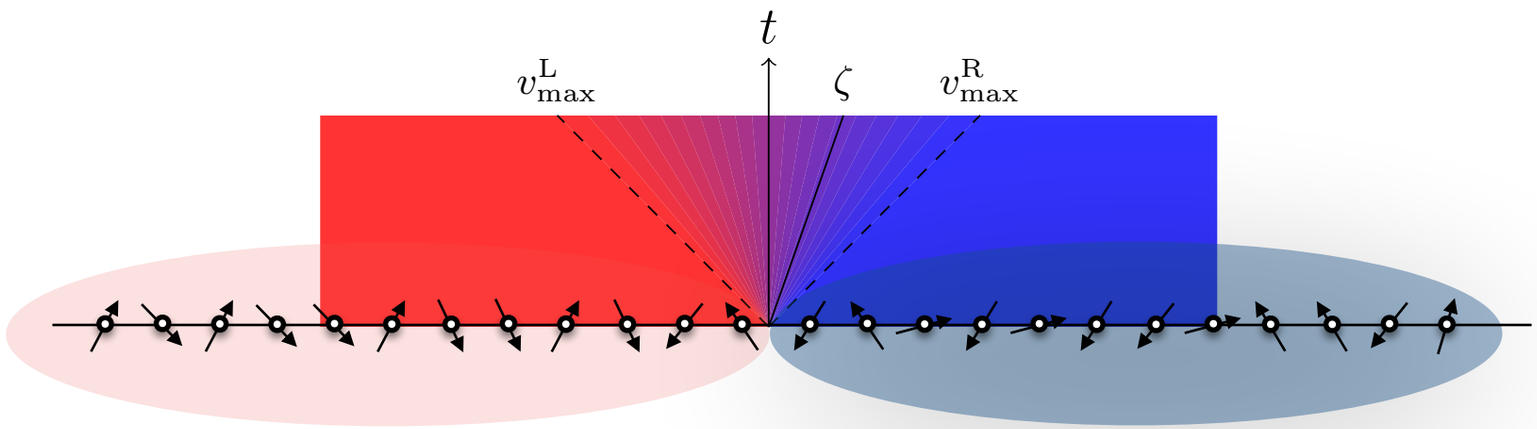


What about sub-ballistic scales (dissipation)?

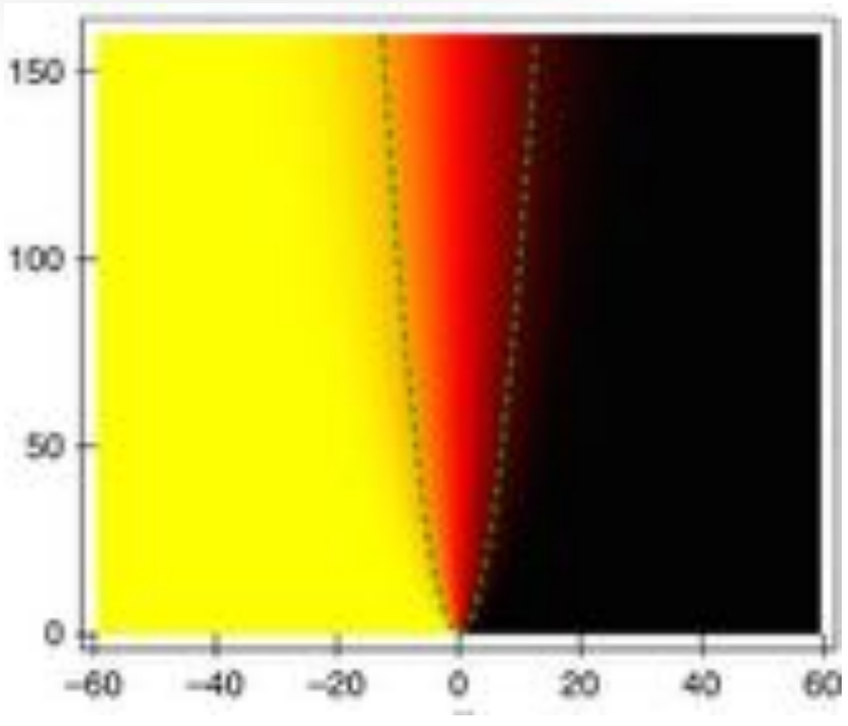
$$\Big(\partial_t + v^{\rm eff}[\vartheta_{x,t}]\partial_x\Big)\vartheta_{x,t}(k_{x,t}) = 0$$

$$\partial_t\vartheta_{x,t} + v^{\rm eff}[\vartheta_{x,t}]\partial_x\vartheta_{x,t} = \partial_x\Gamma[\partial_x\vartheta_{x,t}, \vartheta_{x,t}]$$

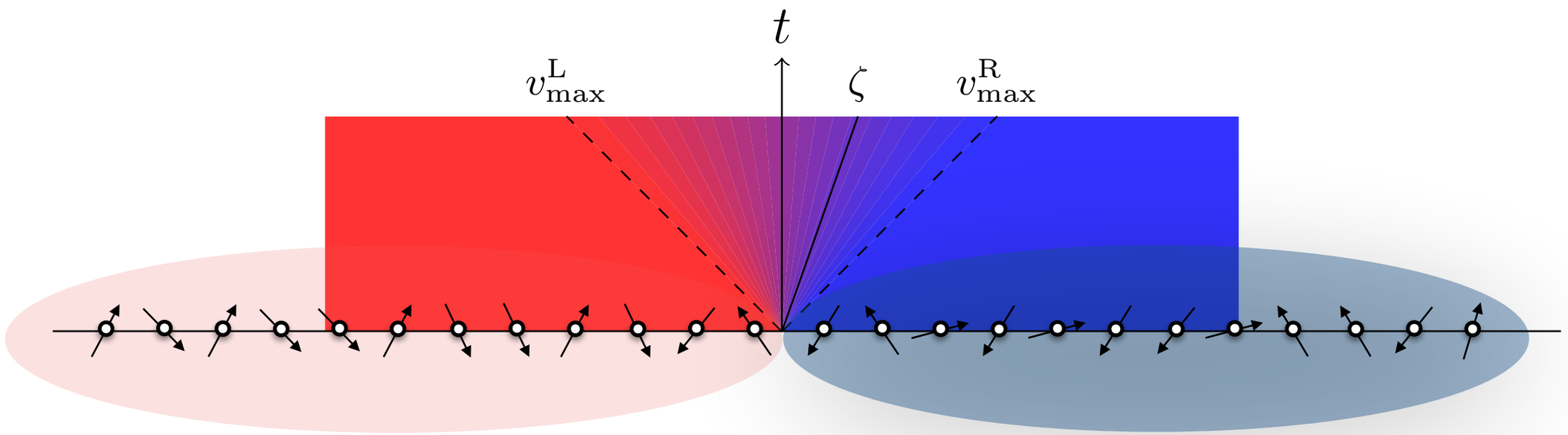
Work in progress with
D. Bernard, B. Doyon



$$\hat{H}_{\rm XXZ} = \sum_{j=1}^{L}\Big(\hat{S}^x_j\hat{S}^x_{j+1} + \hat{S}^y_j\hat{S}^y_{j+1} + \Delta\hat{S}^z_j\hat{S}^z_{j+1}\Big)$$



The diffusion constant at finite temperature



$$\sigma = \lim_{\delta h \to 0} \lim_{t \to \infty} \frac{1}{\delta h} \int dx \langle j(x,t) \rangle \qquad\qquad D(\beta) = \frac{\sigma(\beta)}{\chi(\beta)}$$

$$D(\beta) = \lim_{T \to \infty} \frac{\beta}{\chi(\beta)} \int dx \int_0^T dt \left\langle \hat{j}(x,t) \hat{j}(0,0) \right\rangle$$

$$\lim_{t \to \infty} \frac{\beta}{2} \int dx \left\langle \hat{j}(x,t) \hat{j}(0,0) \right\rangle = \mathcal{D}(\beta)$$

A bound to diffusion constant

PRL 119, 080602 (2017) PHYSICAL REVIEW LETTERS week ending 25 AUGUST 2017

Lower Bounding Diffusion Constant by the Curvature of Drude Weight

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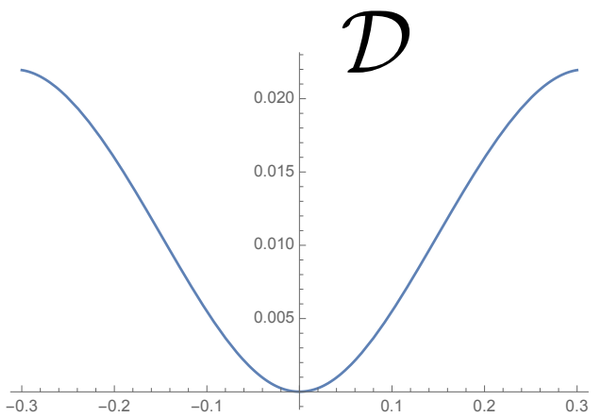
(Received 15 February 2017; revised manuscript received 12 June 2017; published 25 August 2017)



$$\hat{H}_{\mathrm{XXZ}} = \sum_{j=1}^{L} \left(\hat{S}_j^x \hat{S}_{j+1}^x + \hat{S}_j^y \hat{S}_{j+1}^y + \Delta \hat{S}_j^z \hat{S}_{j+1}^z \right)$$

$$\Delta \geq 1$$

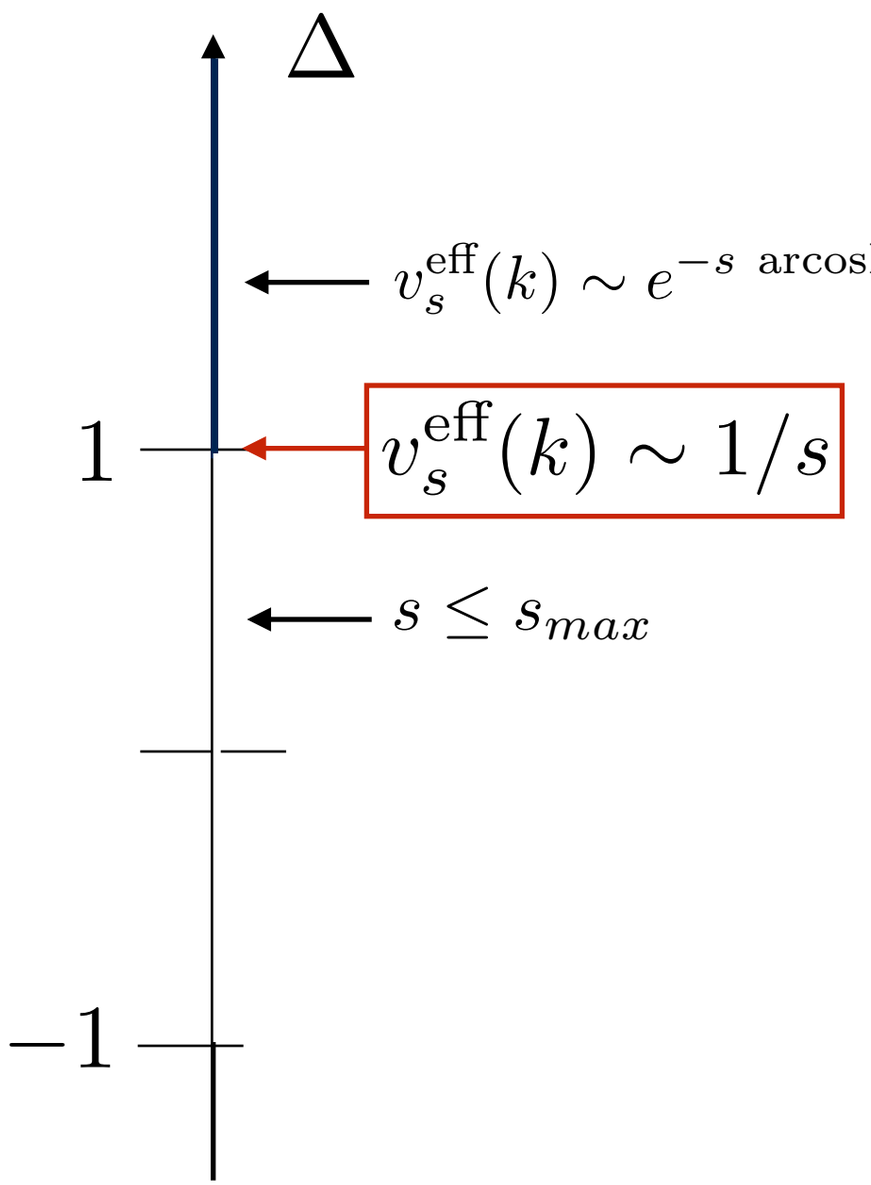
$$D(\beta) \geq \frac{2}{\chi(\beta)} \lim_{t\to\infty} \lim_{L\to\infty} \sum_{m=-1/2}^{1/2} P(m, L)\mathcal{D}(m, \beta)t$$

$$L = 2v_{\mathrm{LR}}t$$

$$D(\beta) \geq C(\beta)\partial_m^2 \mathcal{D}(m, \beta)\Big|_{m=0}$$

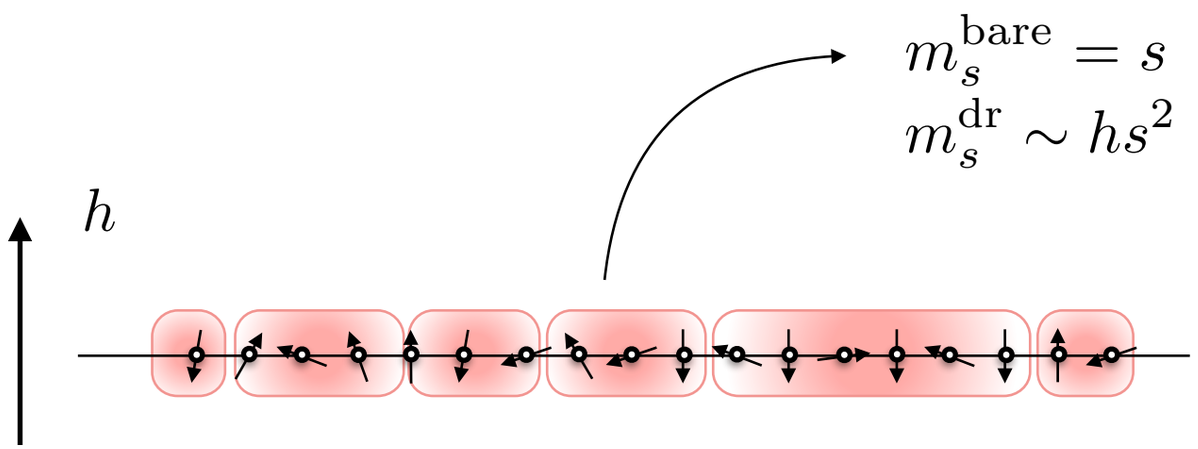
A bound to diffusion constant

$$\hat{H}_{\mathrm{XXZ}} = \sum_{j=1}^{L} \left(\hat{S}_j^x \hat{S}_{j+1}^x + \hat{S}_j^y \hat{S}_{j+1}^y + \Delta \hat{S}_j^z \hat{S}_{j+1}^z \right)$$



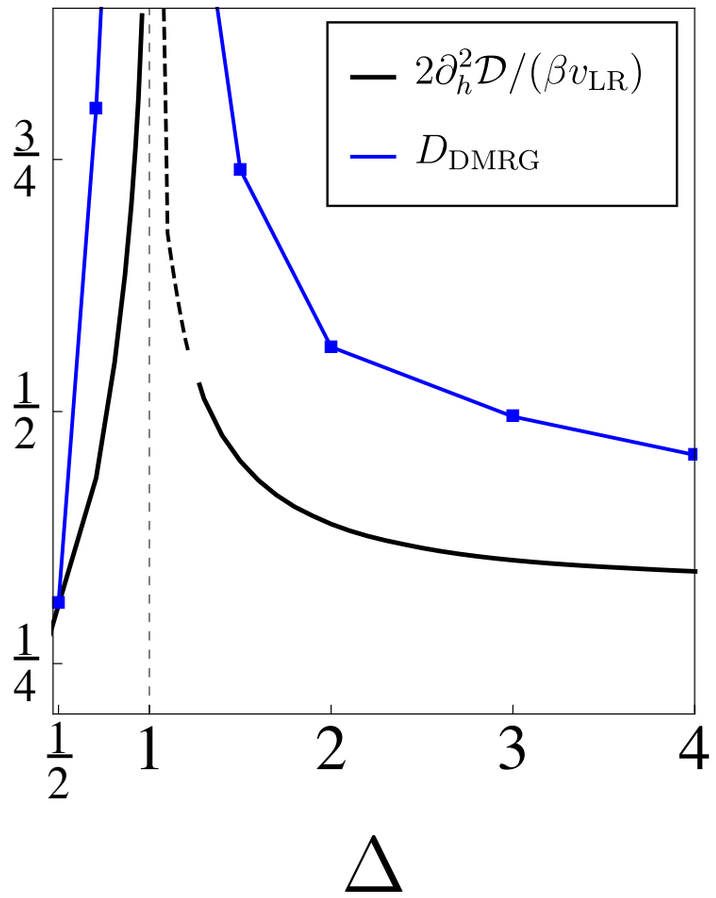
$$D(\beta) \geq C(\beta) \partial_m^2 \mathcal{D}(m, \beta)\Big|_{m=0}$$

$$\mathcal{D} = \frac{\beta}{2} \sum_{s=1}^{\infty} \int dk \vartheta_s(k)(1 - \vartheta_s(k))(v_s^{\mathrm{eff}}(k))^2 (m_s^{\mathrm{dr}}(k))^2$$



Super diffusion at the symmetric point

$\beta = 0$



$$\hat{H}_{\mathrm{XXZ}} = \sum_{j=1}^{L} \left(\hat{S}_j^x \hat{S}_{j+1}^x + \hat{S}_j^y \hat{S}_{j+1}^y + \Delta \hat{S}_j^z \hat{S}_{j+1}^z \right)$$

$\Delta \to 1$

→ $SU(2)$

Isotropic interactions

$D = +\infty$

→ $SU(n)$

→ $SU(2) \otimes SU(2)$ $\quad \hat{H} = -\sum_{j=1}^{L} \sum_{\sigma \in \pm} \hat{c}_{j,\sigma}^\dagger \hat{c}_{j+1,\sigma} + \hat{c}_{j+1,\sigma}^\dagger \hat{c}_{j,\sigma} + 4\mathfrak{u} \sum_{j=1}^{L} \hat{V}_{j,j+1}^{\mathrm{H}}$

→ $SU(2|1)$ $\quad \hat{H} = \hat{P}\Big[-\mathrm{t} \sum_{j,\sigma} \hat{c}_{j,\sigma}^\dagger \hat{c}_{j+1,\sigma} + \hat{c}_{j+1,\sigma}^\dagger \hat{c}_{j,\sigma} \Big] \hat{P} + \mathrm{J} \sum_j \left(\vec{\hat{S}}_j \cdot \vec{\hat{S}}_{j+1} - \frac{1}{4} \hat{n}_j \hat{n}_{j+1} \right)$

Super diffusion at the symmetric point

Article | OPEN

Spin diffusion from an inhomogeneous quench in an integrable system

Marko Ljubotina, Marko Žnidarič & Tomaž Prosen

Nature Communications **8**,
Article number: 16117 (2017)
doi:10.1038/ncomms16117

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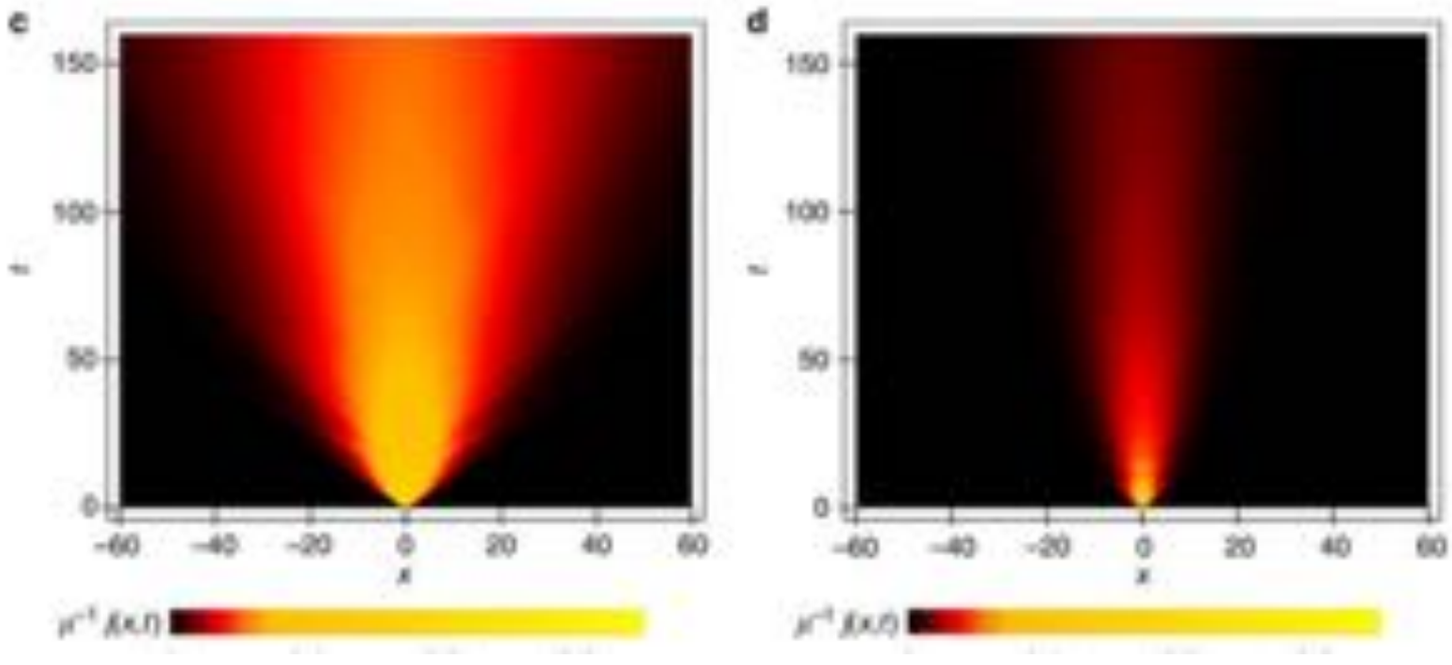
$$\hat{H}_{\mathrm{XXZ}} = \sum_{j=1}^{L} \left(\hat{S}^x_j \hat{S}^x_{j+1} + \hat{S}^y_j \hat{S}^y_{j+1} + \Delta \hat{S}^z_j \hat{S}^z_{j+1} \right)$$

$\Delta = 1$ | $\Delta > 1$



$$\int_0^t j(x=0,t')dt' \sim t^{2/3} \qquad \int_0^t j(x=0,t')dt' \sim t^{1/2}$$

Super diffusion at the symmetric point

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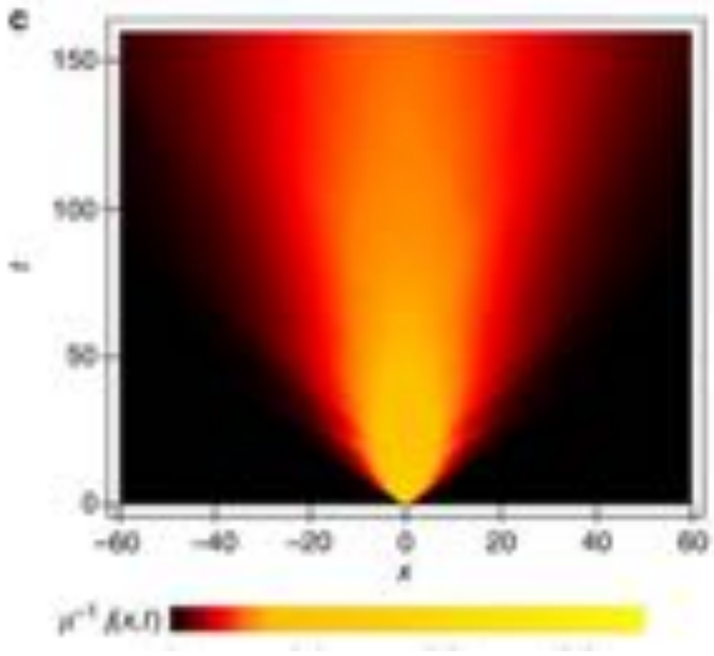
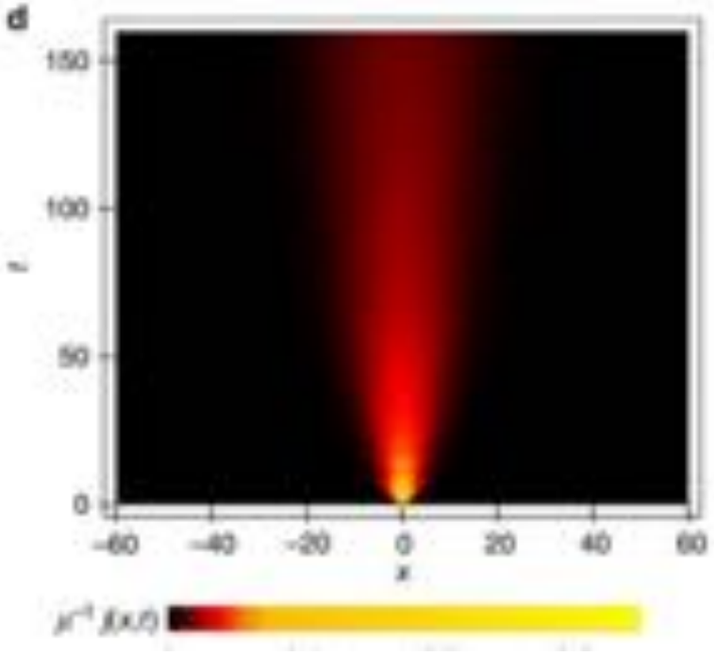
KPZ fluctuations??

$\Delta = 1$



$$\int_0^t j(x=0,t')dt' \sim t^{2/3}$$

$\Delta > 1$



$$\int_0^t j(x=0,t')dt' \sim t^{1/2}$$